Numéro national de thèse (NNT) : 2016SACLS100

UNIVERSITÉ PARIS-SACLAY & PEKING UNIVERSITY

École doctorale de mathématiques Hadamard (EDMH, ED 574) & School of Mathematical sciences

Établissement d'inscription: Université Paris-Sud

Laboratoire d'accueil: Laboratoire de mathématiques d'Orsay, UMR 8628 CNRS

THÈSE DE DOCTORAT EN MATHÉMATIQUES

Spécialité : Mathématiques fondamentales

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CLASSES DE RÉCURRENCE PAR CHAÎNES NON HYPERBOLIQUES DES DIFFÉOMORPHISMES C^1

Date de soutenance : le 24 mai 2016

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Thesis

Classes de récurrence par chaînes non hyperboliques des difféomorphismes C^1

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Doctor of Mathematics

Major in Differential dynamical systems

Peking University, China

Thèse

Pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ PARIS-SUD 11

Discipline: Mathématiques

January, 2016

RÉSUMÉ

La dynamique d'un difféomorphisme d'une variété compacte est essentiellement concentrée sur l'ensemble récurrent par chaînes, qui est partitionné en classes de récurrence par chaînes, disjointes et indécomposables. Le travail de Bonatti et Crovisier [15] montre que, pour les difféomorphismes C^1-génériques, une classe de récurrence par chaînes ou bien est une classe homocline, ou bien ne contient pas de point périodique. Une classe de récurrence par chaînes sans point périodique est appelée classe apériodique.

Il est clair qu'une classe homocline hyperbolique ni contient d'orbite périodique faible ni supporte de mesure non hyperbolique. Cette thèse tente de donner une caractérisation des classes homoclines non hyperboliques en montrant qu'elles contiennent des orbites périodiques faibles ou des mesures ergodiques non hyperboliques. Cette thèse décrit également les décompositions dominées sur les classes apériodiques.

Le premier résultat [69, 71] de cette thèse montre que, pour les difféomorphismes C^1-génériques, si les orbites périodiques contenues dans une classe homocline $H(p)$ ont tous leurs exposants de Lyapunov bornés loin de zéro, alors $H(p)$ doit être (uniformément) hyperbolique. Ceci est dans l'esprit des travaux sur la conjecture de stabilité, mais il y a une différence importante lorsque la classe homocline $H(p)$ n'est pas isolée. Par conséquent, nous devons garantir que des orbites périodiques "faibles", crées par perturbations au voisinage de la classe homocline, sont contenues dans la classe. En ce sens, le problème est de nature "intrinsèque", et l'argument classique de la conjecture de stabilité est impraticable.

Le deuxième résultat [29] de cette thèse prouve une conjecture de Díaz et Gorodetski [41] : pour les difféomorphismes C^1-génériques, si une classe homocline n'est pas hyperbolique, alors elle porte une mesure ergodique non hyperbolique. C'est un travail en collaboration avec C. Cheng, S. Crovisier, S. Gan et D. Yang. Dans la démonstration, nous devons appliquer une technique introduité dans [41], et qui améliore la méthode de [46], pour obtenir une mesure ergodique comme limite d'une suite de mesures périodiques.

Le troisième résultat [70] de cette thèse énonce que, génériquement, une décomposition dominée non-triviale sur une classe apériodique stable au sens de Lyapunov est en fait une décomposition partiellement hyperbolique. Plus précisément, pour les difféomorphismes C^1-génériques, si une classe apériodique stable au sens de Lyapunov a une décomposition dominée non-triviale $E \oplus F$, alors, l'un des deux fibrés est hyperbolique: soit E contracté, soit F dilaté.

Dans les démonstrations des résultats principaux, nous construisons des perturbations qui ne sont pas obtenues directement à partir des lemmes de connexion classiques. En fait, il faut appliquer le lemme de connexion un grand nombre (et même un nombre infini) de fois. Nous expliquons les méthodes de connexions multiples dans le Chapitre 3.

MOTS CLÉS : Classe homocline, classe apériodique, ensemble hyperbolique, ensemble partiellement hyperbolique, décomposition dominée, mesure ergodique, exposant de Lyapunov, généricité, stabilité au sens de Lyapunov.

C^1 微分同胚的非双曲链回复类

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摘　　要

紧致流形上的微分同胚的动力性态主要体现在其链回复集上,链回复集分解为互不相交且不可再分解的链回复类. Bonatti 和 Crovisier [15] 的结果表明,对于 C^1 通有的微分同胚,一个链回复类要么是一个同宿类,要么不含任何周期轨. 一个不含周期轨的链回复类称为一个非周期类.

显而易见,一个双曲的同宿类必不包含弱周期轨也不支撑非双曲测度. 本文试图通过同宿类里面的弱周期轨或者其上的非双曲测度来刻画这个同宿类的非双曲性. 另外,本文也给出了 Lyapunov 稳定的非周期类上的控制分解的一个描述.

本文第一个结果 [69, 71] 表明,对于 C^1 通有的微分同胚, 如果一个同宿类 $H(p)$ 中所有周期轨的 Lyapunov 指数都一致远离 0,那么 $H(p)$ 一定是（一致）双曲的. 这个结果是源于稳定性猜测的一些工作,但是一个显著的不同在于,我们事先并不知道同宿类 $H(p)$ 是孤立集, 从而我们需要保证通过扰动得到的“弱”周期轨要包含在原同宿类中. 从这个意义上讲,这是一个“内在”的问题,因而稳定性猜测的经典讨论在这里就行不通了.

本文第二个结果 [29] 证明了 Díaz 和 Gorodetski [41] 的一个猜测：对于 C^1 通有的微分同胚,如果一个同宿类不双曲,则其必支撑一个非双曲遍历测度. 这是作者与程诚、Sylvain Crovisier、甘少波和杨大伟共同合作的结果. 在证明中,我们用到了 [41] 里面通过取一列周期测度极限得到遍历测度的技巧, 这个技巧是 [46] 中一个方法的推广.

本文的第三个结果 [70] 是,通有情况下,一个 Lyapunov 稳定的非周期类上的非平凡的控制分解实际上是一个部分双曲分解. 确切的说,对于 C^1 通有的微分同胚,如果一个 Lyapunov 稳定的非周期类上有非平凡的控制分解 $E\oplus F$,

那么这两个子丛其中一个是双曲的：或者 E 压缩,或者 F 扩张.

在主要结论的证明中,我们进行了多处扰动,这些扰动不是连接引理的简单应用.实际上,我们需要多次（甚至无穷次）用到连接引理. 我们在第 3 章给出了这些多重连接过程的详细阐述.

关键词: 同宿类,非周期类,双曲集, 部分双曲集,控制分解,遍历测度,Lyapunov 指数,通有性,Lyapunov 稳定性.

Non-hyperbolic chain recurrence classes of C^1 diffeomorphisms

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ABSTRACT

The dynamics of a diffeomorphism of a compact manifold concentrates essentially on the chain recurrent set, which splits into disjoint indecomposable chain recurrence classes. By the work of Bonatti and Crovisier [15], for C^1-generic diffeomorphisms, a chain recurrence class either is a homoclinic class or contains no periodic point. A chain recurrence class without a periodic point is called an aperiodic class.

Obviously, a hyperbolic homoclinic class can neither contain weak periodic orbit or support non-hyperbolic ergodic measure. This thesis attempts to give a characterization of non-hyperbolic homoclinic classes via weak periodic orbits inside or non-hyperbolic ergodic measures supported on it. Also, this thesis gives a description of the dominated splitting on Lyapunov stable aperiodic classes.

The first result [69, 71] of this thesis shows that for C^1-generic diffeomorphisms, if the periodic orbits contained in a homoclinic class $H(p)$ have all their Lyapunov exponents bounded away from 0, then $H(p)$ must be (uniformly) hyperbolic. This is in spirit of the works of the stability conjecture, but with a significant difference that the homoclinic class $H(p)$ is not known isolated in advance. Hence the "weak" periodic orbits created by perturbations near the homoclinic class have to be guaranteed strictly inside the homoclinic class. In this sense the problem is of an "intrinsic" nature, and the classical argument of the stability conjecture does not pass through.

The second result [29] of this thesis proves a conjecture by Díaz and Gorodetski [41]: for C^1-generic diffeomorphisms, if a homoclinic class is not hyperbolic, then there is a non-hyperbolic ergodic measure supported on it. This is a joint work with C. Cheng, S. Crovisier, S. Gan and D. Yang. In the proof, we have to use a technic introduced in [41], which developed the method of [46], to get an ergodic measure by taking the limit of a sequence of periodic measures.

The third result [70] of this thesis states that, generically, a non-trivial dominated splitting over a Lyapunov stable aperiodic class is in fact a partially hyperbolic splitting. To be precise, for C^1-generic diffeomorphisms, if a Lyapunov stable aperiodic class admits a non-trivial dominated splitting $E \oplus F$, then one of the two bundles is hyperbolic: either E is contracted or F is expanded.

In the proofs of the main results, we construct several perturbations which are not simple applications of the connecting lemmas. In fact, one has to apply the connecting lemma several (even infinitely many) times. We will give the detailed explanations of the multi-connecting processes in Chapter 3.

KEY WORDS: Homoclinic class, aperiodic class, hyperbolic set, partially hyperbolic set, dominated splitting, ergodic measure, Lyapunov exponent, genericity, Lyapunov stability.

Contents

Chapter 1

Introduction and statements of the results

1.1 Introduction (Français)

1.1.1 Contexte

L'un des objectifs des systèmes dynamiques est de décrire la plupart des systèmes, c'est-à-dire un ensemble dense, résiduel, ou ouvert et dense. Dans les années 1960, Abraham et Smale [5, 68] ont montré que l'ensemble des *difféomorphismes hyperboliques* (c'est-à-dire satisfaisant l'axiome A et n'ayant pas de cycle) n'est pas dense dans l'espace des systèmes dynamiques. Par conséquent, l'étude de la dynamique loin du cas uniformément hyperbolique est devenue depuis lors un problème majeur pour les dynamiciens. Le défaut d'hyperbolicité peut être caractérisé de différentes façons. Soit M une variété différentiable compacte sans bord de dimension d, et soit $\mathrm{Diff}^r(M)$ l'espace de difféomorphismes C^r de M, pour tout $r \geq 1$.

- Grâce aux contributions de Liao, Mañé, Aoki et Hayashi [53, 54, 6, 50] sur la conjecture de stabilité C^1, il est connu qu'un difféomorphisme non hyperbolique peut être perturbé pour obtenir une orbite périodique non hyperbolique. En particulier, un difféomorphisme C^1-générique (c'est-à-dire un difféomorphisme appartenant à un sous-ensemble G_δ dense de $\mathrm{Diff}^1(M)$) qui n'est pas hyperbolique a des orbites périodiques arbitrairement "faibles".
- La théorie de Pesin [60] affaiblit la notion d'hyperbolicité (en hyperbolicité non uniforme), et donne une approche possible pour caractériser le comportement non hyperbolique par les mesures invariantes.
- Une autre obstruction à l'hyperblicité globale vient des célèbres bifurcations homoclines, dans l'esprit des travaux de Poincaré : il s'agit des tan-

gences homoclines ou des cycles hétérodimensionnels, qui sont également relatifs aux orbites périodiques. Palis a conjecturé [58, 59] qu'un difféomorphisme peut être C^r-approché par un difféomorphisme hyperbolique ou par un difféomorphisme qui présente une bifurcation homocline. Ces bifurcations ont des conséquences dynamiques fortes, et peuvent parfois être renforcé pour aboutir à des obstructions robustes à l'hyperbolicité, voir [20]. Plusieurs progrès ont été obtenus en direction de cette conjecture [24, 33, 34, 37, 39, 64].

La discussion ci-dessus porte sur les obstructions globales à l'hyperbolicité. Nous voulons également étudier le défaut "local" d'yperbolicité, c'est-à-dire le défaut d'hyperbolicité d'une pièce élémentaire de la dynamique. La dynamique est essentiellement concentrée sur l'ensemble des points qui ont des propriété de récurrence, par exemple, *l'ensemble récurrent par chaînes*, qui est partitionné en *classes de récurrence par chaînes*, disjointes et indécomposables, voir [30]. Pour les difféomorphismes C^1-génériques, les orbites périodiques sont denses dans l'ensemble récurrent par chaînes, et la classe de récurrence par chaînes contenant un point périodique p coïncide avec sa *classe homocline*, i.e. avec l'adhérence de l'ensemble des points d'intersection transverse entre les variétés stables et instables de $\mathrm{Orb}(p)$, voir [15, 32]. Ainsi deux classes homoclines ou bien coïncident, ou bien sont disjointes pour les difféomorphismes C^1-génériques. Mais cela n'est pas vrai en général. En outre, les classes homoclines sont généralement en nombre infini, même pour les difféomorphismes C^1-génériques. Une classe de récurrence par chaînes sans point périodique est appelée *classe apériodique*.

Cette thèse caractérise la non-hyperbolicité d'une classe homocline par l'existence d'orbites périodiques faibles contenues dans la classe ou par les mesures ergodiques non hyperboliques qu'elle porte. Nous énoncerons également un résultat portant sur les classes apériodiques et discuterons de la conjecture de Palis.

1.1.2 Hyperbolicité versus orbites périodiques faibles

Nous étudions un problème qui est dans l'esprit des travaux sur la conjecture de stabilité, mais qui a une nature plus "intrinsèque". La conjecture de stabilité formulée par Palis et Smale affirme que, si un difféomorphisme f est structurellement stable, alors il est hyperbolique (c'est-à-dire qu'il satisfait l'axiome A et n'a pas de cycle). Une version plus forte de cette conjecture affirme que, si f est Ω-stable, alors il est hyperbolique. Ces deux conjectures remarquables ont été respectivement résolues par Mañé [55] et Palis [57].

Durant la longue période d'étude des conjectures de stabilité, l'attention s'est concentrée de plus en plus sur les orbites périodiques du difféomorphisme (non perturbé) f et de ses perturbations g. Liao [53] et Mañé [54] ont indépendamment proposé une conjecture (plus précisément un problème sans ébauche de solution), connue sous le nom "conjecture étoile". Elle affirme que si f n'a pas d'orbite périodique non hyperbolique, même après perturbation, alors f est hyperbolique. Cette hypothèse (la condition d'étoile) est clairement plus faible que l'Ω-stabilité. Par conséquent, la conjecture étoile est considérée comme une version forte de la conjecture de stabilité. Elle a été résolue par Aoki et Hayashi [6, 50]. Afin de la comparer précisément avec notre résultat énoncé ci-dessous, nous exposons une version générique de leurs résultats.

Rappelons qu'à chaque mesure invariante ergodique μ d'un difféomorphisme $f \in \mathrm{Diff}^1(M)$, le théorème de Oseledets [56] associe d nombres $\chi_1(\mu, f) \leq \chi_2(\mu, f) \leq \cdots \leq \chi_d(\mu, f)$, tels que, pour μ-presque tout point $x \in M$ et tout $v \in T_xM \setminus \{0\}$, nous avons $\lim_{n\to+\infty} \frac{1}{n} \log \|Df^n(v)\| = \chi_i$ pour un certain $i \in \{1, 2, \cdots, d\}$. Les d nombres $\chi_i(\mu, f)$ sont appelés les *exposants de Lyapunov* de la mesure μ. Nous disons qu'une mesure μ est *hyperbolique*, si tous ses exposants de Lyapunov sont non-nuls. En particulier, si μ est une mesure atomique distribuée uniformeément sur une orbite périodique $\mathrm{Orb}(p)$, les exposants de Lyapunov de μ sont aussi appelés les *exposants de Lyapunov* de l'orbite périodique $\mathrm{Orb}(p)$ (ou du point périodique p).

Théorème 1 ([6, 50])**.** *Un difféomorphisme C^1-générique $f \in \mathrm{Diff}^1(M)$ qui n'est pas hyperbolique possède des orbites périodiques faibles : il existe une suite d'orbites périodiques dont l'un des exposants de Lyapunov converge vers zéro.*

Il est naturel de se demander si le Théorme 1 est vrai ou non pour les ensembles élémentaires d'un difféomorphisme, c'est-à-dire pour ses classes homoclines. Rappelons que deux points périodiques hyperboliques p et q sont *homocliniquement reliés*, si $W^u(\mathrm{Orb}(p))$ a des intersections transversales non vides avec $W^s(\mathrm{Orb}(q))$ et vice versa. La classe homocline d'un point périodique hyperbolique p est aussi égale à l'adhérence de l'ensemble des points périodiques hyperboliques qui lui sont homocliniquement reliés. Voici un problème proposé dans [39].

Problème (Problem 1.8 dans [39])**.** *Considérons un difféomorphisme C^1-générique $f \in \mathrm{Diff}^1(M)$, et une classe homocline $H(p)$. Est-il vrai que*

- *ou bien $H(p)$ est hyperbolique,*
- *ou bien $H(p)$ contient des orbites périodiques faibles qui sont homocliniquement reliées avec p?*

En fait, il est facile de prouver (en suivant la démonstration classique de la conjecture de stabilité) qu'il doit y avoir des orbites périodiques faibles arbitrairement près de la classe $H(p)$. Mais on ne sait pas si elles sont contenues dans la classe. Bien sûr, si la classe homocline $H(p)$ est supposée isolée, alors "être près de" est égale à "être contenu". Mais nous ne savons pas en général si $H(p)$ est isolée (il peut y avoir un nombre infini de classes), même pour les difféomorphismes C^1-génériques. Nous exposons le premier résultat de cette thèse, qui donne une réponse positive à ce problème, voir [69, 71].

Théorème A. *Pour un difféomorphisme C^1-générique $f \in \mathrm{Diff}^1(M)$, si une classe homocline $H(p)$ n'est pas hyperbolique, alors $H(p)$ contient des orbites périodiques faibles qui sont homocliniquement reliées avec p.*

Remarquons que, contrairement aux arguments classiques portant sur la conjecture de stabilité, Le théorème A affirme qu'il doit y avoir des orbites périodiques faibles qui sont non seulement près de mais aussi contenues dans la classe $H(p)$. Ceci est la clé principale du Théorème A. À chaque étape de la démonstration, nous devons garantir que les orbites périodiques créées par les perturbations appartiennent à la classe homocline. En ce sens, nous disons que le problème est de nature "intrinsèque", et les arguments classiques de la conjecture de stabilité sont impraticables.

Dans [55], Mañé a introduit un lemme (Theorem **II.1**) très utile pour obtenir des orbites périodiques faibles sous certaines hypothèses. Son énoncé est très technique et la preuve originale de Mañé est difficile, donc nous ne l'indiquons pas ici. En modifiant de la preuve de Mañé, Bonatti, Gan et Yang ont obtenu un résultat pour les classes homoclines, voir [25]. Nous remarquons ici que, à la différence du Théorème **II.1** de [55] et du résultat de [25], le Théorème A comporte une hypothèse de généricité. Autrement dit, la conclusion du Théorème A est un résultat perturbatif et elle n'est peut-être pas valable pour tous les difféomorphismes.

Nous remarquons que l'hypothèse de généricité est essentielle dans le Théorème A. Un exemple est donné par [66, 28], montre que, en détruisant les fers à cheval hyperboliques d'une famille à paramètres de difféomorphismes de surface, on peut obtenir un difféomorphisme qui appartient à la frontière de l'ensemble des difféomorphismes hyperboliques, et qui possède une classe homocline présentant une tangence homocline. La classe homocline n'est pas hyperbolique en raison de l'existence de la tangence. En outre, tous les exposants de Lyapunov de toutes les mesures ergodiques invariantes sont uniformément bornés loin de zéro, et en particulier, tous les exposants de Lyapunov de toutes les orbites périodiques contenues dans la classe sont uniformément bornés loin de zéro.

Rappelons qu'une *décomposition dominée* $E \oplus F$ sur un ensemble compact invariant Λ est une décomposition Df-invariante de $T_\Lambda M$ telle que la norme de Df le long du sous-fibré E est controlée par celle obtenue long du sous-fibré F. L'ensemble Λ est *partiellement hyperbolique*, si $T_\Lambda M$ possède une dcomposition dominée en trois sous-fibrés $E^s \oplus E^c \oplus E^u$ telle que, les deux sous-fibrés extrémaux sont hyperboliques et le fibré central est neutre (voir la Définition 2.6).

Des exemples de classes homoclines non hyperboliques avec une décomposition dominée ont été trouvés dans [31, 40, 65], mais les classes homoclines dans ces exemples contiennent des orbites périodiques faibles. Pour les difféomorphismes C^2 sur les surfaces, les conclusions de [65] montrent que l'on ne peut pas trouver de classe homocline non hyperbolique avec une décomposition dominée non-triviale et sans orbite périodique faible. Ceci est inconnu dans le cadre de la dynamique C^1. Nous obtenons donc la question suivante.

Question 1. *Existe-t-il un difféomorphisme $f \in \mathrm{Diff}^1(M)$ et une classe homocline non hyperbolique $H(p)$ avec une décomposition dominée non-triviale, tel que, tous les exposants de Lyapunov de toutes les orbites périodiques homocliniquement reliées avec $\mathrm{Orb}(p)$ sont uniformément borné loin de zéro?*

Il y a d'autres conjectures proposant des dichotomies pour la dynamique globale. Rappelons qu'une *tangence homocline* d'un point périodique hyperbolique p est une intersection non transverse entre $W^u(p)$ et $W^s(p)$. Un difféomorphisme présente un *cycle hétérodimensionnel*, s'il y a deux points périodiques hyperboliques *d'indices* (de dimension stable) différentes, tel que $W^s(p) \cap W^u(q) \neq \emptyset$ et $W^s(q) \cap W^u(p) \neq \emptyset$. Il est évident qu'un difféomorphisme avec une tangence ou un cycle hétérodimensionnel n'est pas hyperbolique. Palis [58, 59] a conjecturé que ces deux phénomènes sont les seules obstructions à l'hyperbolicité. Cette conjecture a été résolue par [64] en dimension 2, et pour la dimension supérieure, les travaux de [34, 39, 37] ont donné des réponses partielles : génériquement, les difféomorphismes loin des bifurcations homoclines prśentent certaines formes faibles d'hyperbolicité (hyperbolicité partielle ou essentielle). Basé sur les résultats suivants, Bonatti et Díaz ont conjecturé que l'ensemble des difféomorphismes qui sont hyperboliques et qui présentent un cycle hétérodimensionnel est dense dans l'espace de difféomorphismes, voir [14, 20].

A partir du lemme de Franks [42, 48], nous pouvons perturber des orbites périodiques faibles pour obtenir des orbites périodiques d'indices différents. Mais on ne sait pas si ces orbites périodiques sont encore contenues dans la classe homocline non hyperbolique après la perturbation. Ainsi, nous énonçons la conjecture suivante, qui est une version intrinsèque de la conjecture de Palis pour les classes homoclines.

Conjecture 1 ([14, 17, 20, 34])**.** *Pour un difféomorphisme C^1-générique $f \in \mathrm{Diff}^1(M)$, une classe homocline $H(p)$ qui n'est pas hyperbolique contient un point périodique $q \in H(p)$ d'indice différent de celui de p.*

1.1.3 Hyperbolicité versus mesures non hyperboliques

Il est clair qu'un difféomorphisme possédant une mesure ergodique non hyperbolique ne peut pas être hyperbolique (c'est-à-dire qu'il ne satisfait pas l'axiome A ou bien il présente un cycle). L'inverse n'est pas vrai en général, voir [10, 28], mais nous espérons que c'est le cas des systemes typiques, comme l'on conjecturé Díaz et Gorodetski dans [41].

Conjecture 2 (Conjecture 1 de [41]). *Il existe un sous-ensemble ouvert et dense $\mathcal{U} \subset \mathrm{Diff}^r(M)$ pour $r \geq 1$, tel que, chaque difféomorphisme $f \in \mathcal{U}$ ou bien est hyperbolique, ou bien possède une mesure ergodique invariante non hyperbolique.*

Remarquons que la Conjecture 2 est vraie si nous remplaçons *ouvert et dense* par *dense* seul, ceci est un corollaire direct de la conjecture étoile. Une reformulation légèrement différente et plus faible consiste à considérer les difféomorphismes génériques. Comme Corollaire du Théorème 2 énoncé ci-dessous, la version générique de la Conjecture 2 est vérifiée pour les difféomorphismes *modérés*, i.e. pour ceux dont toutes les classes homoclines sont robustement isolées , voir [41].

Théorème 2 (Théorème 1 de [41]). *Pour les difféomorphismes C^1-génériques, si une classe homocline contient des orbites périodiques d'indices différents, alors elle supporte une mesure ergodique non hyperbolique μ.*

Comme nous l'avons mentionné dans la Section 1.1.2, en général, une classe homocline n'est pas isolée, même pour les difféomorphismes C^1-génériques. Le Théorème 2 indique que, pour résoudre la version générique de la Conjecture 2, il suffit de prouver que chaque difféomorphisme *sauvage* (i.e. qui n'est pas modéré) a une classe homocline qui contient des points périodiques d'indices différents.

On aussi espère que le Théorème 2 se généralise, comme il a été conjectué dans [41].

Conjecture 3 (Conjecture 2 de [41]). *Pour les difféomorphismes C^1-génériques, une classe homocline ou bien est uniformément hyperbolique, ou bien supporte une mesure ergodique invariante non hyperbolique.*

Nous remarquons ici que, la Conjecture 3 n'est pas vraie pour un difféomorphisme général. [66, 28] donne l'exemple d'une classe homocline qui n'est pas hyperbolique en raison de l'existence de tangences homoclines, mais elle est uniformément hyperbolique au sens de la mesure. Plus précisement, tous les exposants de Lyapunov de toutes les mesures invariantes qu'elle supporte sont uniformément bornés loin de zéro. Remarquons qu'une tangence homocline ne persiste pas aux perturbations. Par conséquent, on peut s'attendre à ce que la Conjecture 3 ait une réponse positive. Ceci est le prochain résultat de cette thèse. C'est un travail en collaboration avec C. Cheng, S. Crovisier, S. Gan et D. Yang, voir [29].

Théorème B. *Pour un difféomorphisme C^1-générique $f \in \mathrm{Diff}^1(M)$, une classe homocline $H(p)$ qui n'est pas hyperbolique porte une mesure ergodique non hyperbolique μ.*

Soulignons que le même résultat n'est pas valable pour les classes apériodiques: [18] a construit un ensemble ouvert de difféomorphismes dont les éléments C^1-génériques ont des classes apériodiques qui supportent seulement des mesures ergodiques hyperboliques. Nous n'avons pas de réponse à la Conjecture 2. Une approche possible serait de répondre au problème suivant, voir [2, Conjecture 1], [14, Conjecture 1] ou [36, Section 6.2.Ia].

Conjecture 4. *Un difféomorphisme C^1-générique qui est non hyperbolique possède une classe homocline non hyperbolique.*

Nous commentons à présent les techniques permettant d'obtenir des mesures ergodiques ayant un exposant de Lyapunov nul.

Dans le cas particulier où une classe homocline $H(p)$ a une décomposition partiellement hyperbolique $E^s \oplus E^c \oplus E^u$, avec $\dim(E^c) = 1$, alors, la classe est hyperbolique si et seulement si toutes ses mesures ergodiques ont un exposant de Lyapunov central non nul, de même signe ; chaque mesure hyperbolique est alors approchée par des orbites périodiques hyperboliques de même indice (voir [34] et Proposition 2.45 ci-dessous).

Par conséquent, si $H(p)$ n'est pas hyperbolique, soit elle supporte une mesure ergodique non hyperbolique (comme souhaité), soit elle contient deux orbites périodiques hyperboliques d'indices différents. Dans le second cas, il est possible de construire une mesure non hyperbolique qui mélange les deux orbites périodiques, en utilisant certaines propriétés de pistage. Ceci a été développé dans de nombreux travaux, [46, 41, 22, 13] entre autres. L'hyperbolicité partielle sur un sous-ensemble de la classe est suffisante pour cet argument, voir par exemple le Théorème 2.

La difficulté principale de la preuve du Théorème B est d'obtenir des points périodiques d'indices différents dans une même classe homocline non hyperbolique. Ceci est réalisé sous de nouvelles hypothèses, grâce au Théorème A (qui produit des orbites périodiques faibles) combiné à [17] (pour les transformer en des cycles hétérodimensionnels). Il y a encore des cas où nous ne parvenons pas à obtenir ces points périodiques (voir les discussions dans la Section 4.5 ci-dessous), mais l'existence de mesures non hyperboliques est assurée.

Nous discutons maintenant de différrentes propriétés de ces mesures non hyperboliques.

Plusieurs exposants de Lyapunov nuls Pour les mesures ergodiques non hyperboliques mentionnées ci-dessus nous ne pouvons que garantir l'existence d'un seul exposant de Lyapunov nul. L'exemple de [12] montre qu'il y a des systèmes de fonctions itérées (IFS) qui présentent robustement des mesures ergodiques non hyperboliques avec tous leurs exposants de Lyapunov nuls. Pour les classes homoclines, on peut se poser la question suivante :

Question 2. *Sous quelle hypothèse, existe-t-il une mesure ergodique non hyperbolique supportée sur une classe homocline et ayant plus d'un exposant de Lyapunov nul ?*

Le résultat de [41] dit que, génériquement, si une classe homocline contient des points périodiques d'indices différents, alors, la classe supporte une mesure ergodique non hyperbolique, voir le Théorème 2. Inspiré par ce résultat, on souhaite considérer la question suivante :
Si une classe homocline contient des orbites périodiques d'indices i et j respectivement, où $i < j$, est-ce qu'il existe une mesure ergodique supportée sur la classe homocline, telle que les exposants de Lyapunov $(i+1)$ à j sont tous nuls ?

Évidemment, ce n'est pas vrai si la classe homocline a une décomposition dominée $E \oplus F$ avec $i < \dim(E) < j$. Que se qui se passe-t-il s'il n'y a pas de telle décomposition dominée ? Nous énonçons un travail en collaboration avec J. Zhang [72], qui répond partiellement à la Question 2. Rappelons que pour une décomposition dominée $E \oplus F$, nous appelons $\dim(E)$ son *indice*.

Théorème 3 ([72]). *Pour un difféomorphism C^1-générique $f \in \mathrm{Diff}^1(M)$, considérons un point périodique hyperbolique p. Soit $H(p)$ une classe homocline satisfaisant les propriétés suivantes :*

- *la classe homocline $H(p)$ contient deux points périodiques hyperboliques q_1, q_2 d'indices i et j respectivement, où $i < j$,*
- *la classe homocline $H(p)$ n'a pas de décomposition dominée d'indice k pour tout $k = i+1, \cdots j-1$.*

Alors il y a une mesure ergodique μ satisfaisant dont le $k^{\text{ème}}$ exposant de Lyapunov est nul pour tout $k = i+1, \cdots j$, et son support est $H(p)$.

Nous ne donnerons pas la preuve du Théorème 3 dans la thèse.

Support Le Théorème B et le Théorème 2 permettent d'obtenir des mesures ergodiques non hyperboliques supportées sur une classe homocline. On s'intéresse à la question suivante:
Est-ce que les supports des mesures ergodiques non hyperboliques peuvent être la classe homocline tout entière ?

La réponse est connue dans certains cas, voir par exemple le Théorème 3 ci-dessus. Un résultat précédent [22] avait montré que, pour les difféomorphismes

C^1-génériques, si une classe homocline a une décomposition dominée en trois fibrés $E \oplus F \oplus G$ avec $\dim(F) = 1$, et si elle contient deux orbites périodique d'indices respectifs $\dim(E)$ et $\dim(E) + 1$, alors il y a une mesure ergodique non hyperbolique μ, dont l'exposant de Lyapunov le long de le fibré F est nul et dont le support est la classe homocline toute entière .

Nous donnerons un résultat plus général que celui proposé par [22] en la Section 5.4 : il dit que, génériquement, si une classe homocline contient des points périodiques d'indices différents, alors elle porte une mesure ergodique non hyperbolique, dont le support est la classe homocline tout entière. Ce résultat généralise également le Théorème 2.

Existence robuste On voudrait aussi savoir si l'existence de mesures ergodiques non hyperboliques est une propriété robuste, comme pour la Conjecture 2. Une question similaire a été soulevée dans [12] pour les mesures ergodiques comportant plus d'un exposant de Lyapunov nul.

Question 3. *Existe-t-il un ensemble ouvert $\mathcal{U}$ de difféomorphismes possédant une mesure ergodique ayant plus d'un exposant de Lyapunov nul ?*

Autres questions D'autres questions concernant des mesures ergodiques non hyperboliques supportées par des classes homoclines peuvent être posées. Bonatti a proposé les deux suivantes :

(1) *Est-ce qu'il existe une telle mesure d'entropie positive?*

Ceci a été obtenu dans [13] pour les classes homoclines contenant des points périodiques d'indices différentes.

(2) *Considérons l'adhérence de l'ensemble des mesures ergodiques dont le $i^{\text{ème}}$ exposant de Lyapunov s'annule. Est-ce un ensemble convexe ?*

1.1.4 Décompositions dominées sur les classes apériodiques stables au sens de Lyapunov

On souhaite aussi étudier les classes de récurrence par chaînes qui admettent une base de voisinages attractifs, par exemple [7, 19, 37, 62, 80]. Rappelons qu'un ensemble compact invariant K est *stable au sens de Lyapunov*, si pour chaque voisinage U de K, il y a un autre voisinage V, tel que $f^n(V) \subset U$ pour tout $n \geq 0$. Et K est appelé *bi-stable au sens de Lyapunov*, si il est stable au sens de Lyapunov pour f et f^{-1}.

Par [34, 39, 81], pour les difféomorphismes C^1-génériques loin des bifurcations homoclines (ou des tangences homoclines seulement), une classe apériodique est partiellement hyperbolique avec un fibré central de dimension 1. Les travaux [37, 80] ont prouvé que, pour un difféomorphisme C^1-générique loin des bifurcations homoclines, une classe apériodique ne peut pas être stable au sens de Lyapunov. Mais dans [19], il est montré qu'il y a un ensemble ouvert

$\mathcal{U} \subset \mathrm{Diff}^1(M)$, tel que, les difféomorphismes génériques $f \in \mathcal{U}$ ont un nombre infini de classes apériodiques qui sont bi-stables au sens de Lyapunov, et qui n'admettent pas de décomposition dominée. Dans [62], Potrie a prouvé que, pour les difféomorphismes C^1-génériques, si une classe homocline est bi-stable au sens de Lyapunov, alors elle admet une décomposition dominée non-triviale, et sous certaines hypothèses, elle coïncide avec la variété tout entière. Il y a aussi beaucoup d'autres résultats pour des classes homoclines stables au sens de Lyapunov dans [62, 7].

Nous exposons ici une conjecture sur les classes apériodiques par Crovisier dans son rapport pour l'ICM 2015, qui implique la non-existence de classe apériodique pour les difféomorphismes C^1-génériques loin des bifurcations homoclines (ou des tangences homoclines).

Conjecture 5 ([36]). *Soit $f \in \mathrm{Diff}^1(M)$ un difféomorphisme C^1-générique et Λ une classe apériodique de f. Soit $E^s \oplus E^c \oplus E^u$ la décomposition dominée sur Λ telle que le fibré E^s est le sous-fibré contracté maximal et le fibré E^u est le sous-fibré dilaté maximal. Alors le fibré E^c a une dimension au moins égale à deux et n'admet aucune décomposition dominée non-triviale.*

Le Théorème suivant est le troisième résultat de cette thèse, voir [70]. Il affirme qu'une décomposition dominée non-triviale sur une classe apériodique stable au sens de Lyapunov est en fait une décomposition partiellement hyperbolique pour les difféomorphismes C^1-génériques. Ceci donne une réponse partielle à la Conjecture 5.

Théorème C. *Pour un difféomorphisme C^1-générique $f \in \mathrm{Diff}^1(M)$, si une classe apériodique Λ stable au sens de Lyapunov de f admet une décomposition dominée $T_\Lambda M = E \oplus F$, alors, ou bien le fibré E est contracté, ou bien le fibré F est dilaté.*

Comme conséquence du Théorème C, pour les difféomorphismes C^1-génériques, si on considére la décomposition dominée $E^s \oplus E^c \oplus E^u$ sur une classe apériodique stable au sens de Lyapunov, pour laquelle le fibré E^s est le sous-fibré contracté maximal et le fibré E^u est le sous-fibré dilaté maximal, alors le sous-fibré E^c n'admet aucune décomposition dominée non-triviale. En outre, par les arguments de [36, 37], en utilisant le "modèle central" construit dans [33], on sait que la dimension du sous-fibré E^c est au moins de deux. Par conséquence, la Conjecture 5 est valable pour les classes apériodique stables au sens de Lyapunov.

Nous remarquons que l'énoncé du Théorème C n'est pas valable pour les classes homoclines, ce qui montre que l'apériodicité est une hypothèse essentielle. En effet, [27] a construit un difféomorphisme robustement transitif de $\mathbb{T}^4$, pour lequel la variété entière est une classe homocline stable au sens de Lyapunov, et n'admettant qu'une décomposition dominée $E \oplus F$ avec

$\dim(E) = \dim(F) = 2$. De plus, il y a des orbites périodiques de toutes les indices possibles, ce qui implique que ni E est contracté, ni F est dilaté.

Avec le Théorème C, on peut obtenir qu'un (et seulement un) des deux sous-fibrés E et F est hyperbolique, mais on ne sait pas lequelle. Pour un point x contenu dans une classe de récurrence par chaînes stable au sens de Lyapunov, l'ensemble instable $W^u(x) = \{y \in M : \lim_{n\to+\infty} d(f^{-n}(x), f^{-n}(y)) = 0\}$ est aussi contenu dans la classe. Par conséquent, si le sous-fibré F est dilaté, par [51], la classe apériodique stable au sens de Lyapunov est feuilletée par les variétés instables qui sont tangentes au sous-fibré F ; par conséquent, elle ne peut pas être minimale. Nous conjecturons que ce cas n'est pas possible.

Conjecture 6. *Pour un difféomorphisme C^1-générique $f \in \mathrm{Diff}^1(M)$, si une classe apériodique Λ stable au sens de Lyapunov de f admet une décomposition dominée $T_\Lambda M = E \oplus F$, alors le fibré E est contracté.*

Bonatti et Shinohara ont un programme pour construire des classes apériodiques avec une décomposition dominée non-triviale. Puisque les classes apériodiques ne sont pas isolées, ces exemples ne sont pas facile à construire. Même la construction des classes homoclines stables au sens de Lyapunov non isolées est délicate, voir [26].

En fait, chaque classe apériodique bi-stable au sens de Lyapunov construité dans [21] est un ensemble de Cantor minimal, et elle n'admet aucune décomposition dominée non-triviale. Par conséquent, nous énonçons une seconde conjecture, qui est évidemment vraie si la Conjecture 6 est vraie.

Conjecture 7. *Pour un difféomorphisme C^1-générique $f \in \mathrm{Diff}^1(M)$, si une classe apériodique Λ de f est bi-stable au sens de Lyapunov, alors elle n'admet pas de décomposition dominée non-triviale.*

Un résultat [82] lié à la Conjecture 7, affirme que, si un difféomorphisme est minimal (la variété entière est alors une classe apériodique stable au sens de Lyapunov), alors il n'admet aucune décomposition dominée non-triviale.

1.2 Introduction (Chinese)

1.2.1 Backgrounds (Chinese)

研究动力系统的目标之一是描述大多数系统,即：动力系统空间的稠密子集、剩余子集或是开稠子集.十九世纪六十年代, Abraham和Smale [5, 68] 发现,**双曲微分同胚**（即满足公理 A 和无环条件）在微分同胚空间中不稠密. 从那以后,一致双曲之外的动力系统就成为学者们研究的主要课题之一.非双曲性可以通过不同方式来刻画. 记 M 为一个 d 维紧致光滑无边流形,记 $\mathrm{Diff}^r(M)$ 为 M 上的 C^r 微分同胚空间,其中 $r \geq 1$.

– 通过廖山涛,Mañé,Aoki以及Hayashi [53, 54, 6, 50] 在 C^1 稳定性猜测方面的成果, 人们知道了任何一个非双曲的微分同胚可以通过任意小扰动得到非双曲周期点. 特别的,一个 C^1 通有微分同胚（即 $\mathrm{Diff}^1(M)$ 中一个稠密 G_δ 集中的微分同胚）如果不是双曲的,那么一定有任意“弱”的周期轨.

– Pesin's理论 [60] 给出一种弱化的双曲性（非一致双曲）,从而提供了通过不变测度来刻画非双曲性态的一种可能的方法.

– 双曲性的另外一个整体阻碍是源于 Poincaré 的著名的同宿扰动：同宿切或者异维环,二者仍然是跟周期轨有关. Palis [58, 59] 猜测,任何微分同胚可以被双曲系统或者含有同宿切或异维环的系统 C^r 逼近. 同宿切和异维环有着非常丰富的动力性态,而且在一些情形下可以具有持续性,见 [20]. 在Palis猜测方面有一大批的前沿工作,见 [24, 33, 34, 37, 39, 64].

上述讨论是关于双曲性的整体阻碍.人们也会研究“局部的”非双曲性,也就是动力系统一个基本块上的双曲性. 微分同胚的动力性态最终是体现在那些具有一定回复性的点的集合上,比如**链回复集**, 其又可以被分解为互不相交的不可再分解的**链回复类**,见 [30]. 对于 C^1 通有微分同胚,周期轨在链回复集中是稠密的,且任何一个包含一个周期点 p 的链回复类就是其**同宿类**：p 点

轨道的稳定流形和不稳定流形横截交点的闭包,见 [15, 32]. 从而对于 C^1 通有微分同胚,两个同宿类或者重合或者互不相交,而这点对于一般的微分同胚是不对的. 一般情况下,即使是 C^1 通有微分同胚,也会包含无穷多个同宿类. 我们称一个不包含周期点的链回复类为**非周期类**.

本文将通过同宿类包含的弱周期轨或者其支撑的非双曲遍历测度来刻画其非双曲性. 另外,本文也给出了关于非周期类的一个结果,以及围绕 Palis 猜测做一些讨论.

1.2.2 Hyperbolicity versus weak periodic orbits (Chinese)

我们研究一个秉承稳定性猜测,而又具有一种“内在”性质的问题. Palis 和 Smale 提出的稳定性猜测断言,如果一个微分同胚 f 是结构稳定的,那么其是双曲的（即满足公理 A 和无环条件）. 一个更强的形式是如果 f 是 $\Omega-$稳定的,那么其是双曲的. 这两个重要的猜测分别由 Mañé [55] 和 Palis [57] 解决.

在研究稳定性猜测的漫长过程中,（未经扰动的）微分同胚 f 以及其扰动后的微分同胚 g 的周期轨越来越吸引人们的注意力. 廖山涛 [53] 和 Mañé [54] 独立的提出一个猜测（确切的说是一个没有倾向性答案的问题）,即星号猜测：如果微分同胚 f 持续的没有非双曲周期轨,那么其是双曲的. 作为假设,星号条件明显的比 $\Omega-$稳定要弱,从而星号假设也被认为是稳定性猜测的另一（强）形式. 星号猜测由 Aoki 和 Hayashi [6, 50] 解决. 为了跟本文的结果进行对比,我们将他们的结果陈述为通有形式.

回顾一下,由 Oseledets 定理 [56],对于微分同胚 $f \in \mathrm{Diff}^1(M)$ 的一个不变遍历测度 μ, 存在 d 个常数 $\chi_1(\mu,f) \le \chi_2(\mu,f) \le \cdots \le \chi_d(\mu,f)$,满足,对于 μ 几乎处处的点 $x \in M$ 以及任意向量 $v \in T_xM \setminus \{0\}$, 存在 $i \in \{1,2,\cdots,d\}$,使得 $\lim_{n\to+\infty}\frac{1}{n}\log\|Df^n(v)\| = \chi_i$ 成立. 这 d 个常数 $\chi_i(\mu,f)$ 称为 μ 的 **Lyapunov 指数**. 如果 μ的所有 Lyapunov 指数都非零,那么称 μ **双曲**. 特别的,如果 μ 是平均分布在一个周期轨道 $\mathrm{Orb}(p)$ 上的原子测度, 那么 μ 的 Lyapunov 指数也称为周期轨 $\mathrm{Orb}(p)$ （或者周期点 p ）的 **Lyapunov 指数**.

定理 1 ([6, 50]) 对于通有的微分同胚 $f \in \mathrm{Diff}^1(M)$,如果 f 不双曲,那么必有弱周期轨:存在一列周期轨使得其某个 Lyapunov 指数趋于 0.

人们自然想考虑定理 1 对于一个微分同胚的基本块,也就是同宿类,是否仍然成立. 回顾,两个双曲周期点 p 和 q 称为**同宿相关**的,如果 $W^u(\mathrm{Orb}(p))$ 与 $W^s(\mathrm{Orb}(q))$ 有非空的横截交点,反之亦然. 一个双曲周期点 p 的同宿类也等于所有与其同宿相关的周期点构成的集合的闭包.下面是文献 [39] 提出的一个问题.

问题 ([39] **中**Problem 1.8) 对于通有的微分同胚 $f \in \mathrm{Diff}^1(M)$ 以及任何同宿类 $H(p)$,是否有:

- 或者 $H(p)$ 双曲,
- 或者 $H(p)$ 包含与 p 同宿相关的弱周期轨?

实际上,可以直接证明(用稳定性猜测的经典证明方法),同宿类 $H(p)$ 附近一定存在弱周期轨.但是我们不知道弱周期轨是否包含在其中. 当然,如果假设同宿类 $H(p)$ 是孤立的,那么“在附近”就等于“在其中”. 这里的关键点在于即使对于 C^1 通有的微分同胚,一个同宿类 $H(p)$ 也不是先验孤立的(可能有无穷多个类). 下面我们陈述本文的第一个结果,这给出来上述问题的一个肯定回答,见 [69, 71].

定理 A. 对于通有的微分同胚 $f \in \mathrm{Diff}^1(M)$,如果一个同宿类 $H(p)$ 不双曲,那么 $H(p)$ 包含与 p 同宿相关的弱周期轨.

注意到,与稳定性猜测的经典讨论不同的是,这里定理 A 断言一定有弱周期轨,不仅仅是在 $H(p)$ 附近,而且实际上是在其中. 这是定理 A 的关键所在.在证明的每个步骤中,我们要保证通过扰动得到的周期轨包含在同宿类中. 从这个意义上讲,这个问题是具有“内在的”性质,而稳定性猜测的经典讨论在这里行不通.

文献 [55] 中,Mañé 引入一个非常有用的引理(Theorem **II.1**),在某些假设下来得到弱周期轨. 其陈述非常技巧化,而且 Mañé 的原始证明非常复杂,我们不在本文中给出陈述. 基于对 Mañé 的证明的一个修正,Bonatti,甘少波和

杨大伟得到一个关于同宿类的结果,见 [25]. 在这里我们指出,与 [55] 中 Theorem **II.1** 和 [25] 的结果不同,定理 A 中有通有假设. 也就是说定理 A 是一个扰动结果,可能并不是对所有的微分同胚都成立.

我们指出,定理 A 中的通有假设是关键的. 文献 [66, 28] 构造了一个例子,通过破坏曲面上一个微分同胚参数族的双曲的马蹄,得到双曲系统集合边界上的一个微分同胚. 此系统的一个同宿类中含有同宿切.由于同宿切的存在,可知这个同宿类不双曲. 然而,此同宿类支撑的所有遍历测度的 Lyapunov 指数是一致远离零的,特别的,同宿类中所有周期轨的 Lyapunov 指数是一致远离零的.

回顾一下,一个紧致不变集 Λ 上的一个**控制分解** $E \oplus F$ 是 $T_\Lambda M$ 上的一个 Df 不变的分解, 而且 Df 沿子丛 E 上的模被其沿子丛 F 上的模控制. 称 Λ 是**部分双曲**的,如果 $T_\Lambda M$ 有分解为三个子丛 $E^s \oplus E^c \oplus E^u$的控制分解, 满足子丛 E^s 和 E^u 是双曲的,见 Definition 2.6. 具有控制分解的非双曲同宿类的例子可见 [31, 40, 65],而这些例子中的同宿类都是包含弱周期轨的. 由 [65] 的结论可知,对于曲面上 C^2 微分同胚,没有不含弱周期轨而具有控制分解的非双曲同宿类. 这一点对于 C^1 系统是未知的.我们有下面一个问题.

问题 1 *是否存在一个微分同胚 $f \in \mathrm{Diff}^1(M)$ 以及一个非双曲同宿类 $H(p)$,满足 $H(p)$ 具有非平凡控制分解,而跟 $\mathrm{Orb}(p)$ 同宿相关的所有周期轨的 Lyapunov 指数是一致远离零的?*

对于动力系统整体分类,还有其他的猜测.回顾,一个双曲周期点 p 的**同宿切**,是 $W^u(p)$ 和 $W^s(p)$ 的非横截交点. 称一个微分同胚含有**异维环**,如果存在两个不同**指标**(稳定维数)的周期点 p 和 q, 满足 $W^s(p) \cap W^u(q) \neq \emptyset$ 以及 $W^s(q) \cap W^u(p) \neq \emptyset$. 易见任何含有同宿切和异维环的微分同胚都是不双曲的.Palis [58, 59] 猜测这两种现象是双曲性的唯一阻碍. 这个猜测的二维情形由 [64] 解决,而在高维情形,文献 [34, 39, 37] 取得了很大的进展:通有情况下,远离同宿切和异维环的系统具有某些弱双曲性(部分双曲或者本质双曲). 基于以后的工作,Bonatti 和 Díaz猜测双曲系统和具有异维环的系统构成的集合在微分动力系统空间中稠密,见 [14, 20].

由Franks引理 [42, 48],我们可以通过扰动改变弱周期轨的指标,但是我们不知道扰动后得到的周期轨是否还在原同宿类中. 我们有下面的猜测,可以将其看做 Palis 猜测在同宿类上的一种内在形式.

猜测 1 ([14, 17, 20, 34]) *对于通有微分同胚 $f \in \mathrm{Diff}^1(M)$,如果一个同宿类 $H(p)$ 不双曲,那么存在指标不同于 p 的周期点 $q \in H(p)$.*

1.2.3 Hyperbolicity versus non-hyperbolic measures (Chinese)

显然,如果一个微分同胚有一个非双曲遍历测度,那么其一定不双曲(即满足公理 A 和无环条件). 反之则不然,见 [10, 28],但是正如 Díaz 和 Gorodetski [41]的猜测,我们期望对于典型的系统,反之亦然,

猜测 2 ([41] 中 Conjecture 1) *对于 $r \geq 1$,存在 $\mathrm{Diff}^r(M)$ 中的开稠子集 $\mathcal{U}$,使得任意微分同胚 $f \in \mathcal{U}$, 或者双曲,或者有非双曲遍历测度.*

注意到,如果我们将**开稠**替换成**稠**,那么猜测 2 自然成立, 这是星号猜测的一个直接推论.一个稍微弱一点的形式,是考虑通有的微分同胚.作为下述定理 2 的一个推论, 对于**保守**微分同胚:其所有的同宿类是持续孤立的, 猜测 2 的通有形式是成立的,见 [41].

定理 2 ([41] 中 Theorem 1) *对于通有微分同胚 $f \in \mathrm{Diff}^1(M)$,如果一个同宿类包含不同指标周期轨, 那么其上必支撑一个非双曲遍历测度 μ.*

正如我们在第 1.2.2 节中所提,一般来说,即使是对 C^1 通有的微分同胚,同宿类往往不是孤立的. 定理 2 表明,要解决猜测 2, 只需证明任何一个非保守微分同胚都有一个同宿类包含不同指标的周期轨.

我们期望定理 2 的一般形式是成立的,这也是 [41] 的猜测.

猜测 3 ([41] 中 Conjecture 2) *对于通有微分同胚 $f \in \mathrm{Diff}^1(M)$,任何一个同宿类或者双曲,或者支持一个非双曲遍历测度.*

我们指出,猜测 3 对于一般的微分同胚是不对的. 文献 [66, 28] 中的例子是一个含有同宿切从而不双曲的同宿类,而其在测度意义上讲是一致双曲的.

确切的说,这个同宿类支撑的所有不变测度的所有 Lyapunov 指数是一致远离零的. 然而,同宿切在扰动下会被破坏,从而我们期望给出猜测 3 一个肯定的答案. 这也是本文的第二个结果,这是作者与程诚、S. Crovisier、甘少波和杨大伟合作的一个结果,见 [29].

定理 B. *对于通有微分同胚 $f \in \mathrm{Diff}^1(M)$,一个不双曲同宿类 $H(p)$ 必支撑一个非双曲遍历测度 μ.*

我们强调一下,同样的陈述对于非周期类是不成立的：文献 [18] 构造了微分动力系统空间的一个开稠集合,满足对于其中 C^1 通有的元素,其非周期类只支撑双曲遍历测度. 我们不能给出猜测 2 一个回答. 一个可能的尝试是回答下述猜测,见 [2, Conjecture 1], [14, Conjecture 1] 或者[36, Section 6.2.Ia].

猜测 4 *任何 C^1 通有微分同胚如果不双曲,则必包含非双曲同宿类.*

我们来陈述一下得到非双曲遍历测度的一些技巧.

在一些特殊情形下,比如同宿类 $H(p)$ 有中心一维部分双曲分解 $E^s \oplus E^c \oplus E^u$, 此时同宿类双曲当且仅当其支撑的所有遍历测度中心指数非零且符号一样. 任何一个双曲测度可以被同宿类中相同指标的周期轨逼近,见 [34] 以及本文中 Proposition 2.45. 从而如果 $H(p)$ 不双曲,则要么其支撑一个非双曲测度（正如我们所期望的）,要么其包含不同指标周期轨. 在第二种情形下,一个可能的方法是用某种跟踪性质,通过混合两个不同指标周期轨来构造非双曲测度. 这种技巧在很多工作中用到,其中比如 [46, 41, 22, 13]. 实际上,对于这种讨论,同宿类的一个子集上的部分双曲分解就足够了,比如定理 2.

证明定理 B 的主要困难是在同宿类中得到不同指标周期轨. 结合定理 A（得到弱周期轨）以及 [17] 中的结果（将弱周期轨扰出异维环）, 我们可以在一些新的设定下得到不同指标周期轨. 在一些情形下,我们仍然不能得到这样的周期轨（见本文 Section 4.5 中的讨论）,但是非双曲遍历测度的存在性可以保证.

我们再来讨论非双曲遍历测度的更多的性质.

多个退化 Lyapunov 指数 上文中提到的非双曲遍历测度只能保证有一个退化的 Lyapunov 指数. 而文献 [12] 中的例子表明,存在持续的迭代函数系统（IFS）支撑所有 Lyapunov 指数都退化的非双曲遍历测度. 同样对于同宿类,我们可以提出类似的问题：

问题 2 *在何种条件下,存在一个支撑在同宿类上的多个 Lyapunov 指数退化的非双曲遍历测度?*

文献 [41]的结果表明,通有条件下,如果一个同宿类包含不同指标周期轨,那么其必支撑一个非双曲遍历测度,见定理 2. 基于此结果,我们可以考虑下述问题：

如果一个同宿类包含指标分别为 i 和 j 的周期轨,其中 $i<j$, 是否存在非双曲遍历测度,其第 $i+1$ 直到第 j 个 Lyapunov 指数均退化?

显然,如果同宿类有控制分解 $E\oplus F$,满足 $i<\dim(E)<j$,那么上述问题是不成立的. 如果同宿类控制分解会怎样呢？我们来陈述本文作者与张金华最近的一个合作结果 [72], 这给出了问题 2 的一个部分解答. 回顾一下,我们称 $\dim(E)$ 为一个控制分解 $E\oplus F$ 的**指标**.

定理 3 ([72]) *对于通有微分同胚 $f\in \mathrm{Diff}^1(M)$,考虑一个双曲周期点 p. 假设其同宿类 $H(p)$ 满足以下性质:*

- *同宿类 $H(p)$ 包含指标分别为 i 和 j 的周期轨 q_1,q_2,其中 $i<j$;*

- *对任意 $k=i+1,\cdots,j-1$,同宿类 $H(p)$ 包没有指标为 k 的控制分解.*

那么同宿类 $H(p)$ 支撑一个遍历测度 μ,其支集为整个同宿类 $H(p)$,且其第 $i+1$ 直到第 j 个 Lyapunov 指数均退化.

本文中不给出定理 3 的证明.

支集 定理 B和定理 2 能得到支撑在同宿类上的非双曲遍历测度. 我们自然可以问以下关于其支集的问题：

支撑在同宿类上的非双曲遍历测度的支集能是整个同宿类吗?

在一些情况下我们可以给出肯定答案,比如定理 3. 文献 [22] 证明了,对于 C^1 同样微分同胚,如果一个同宿类有中心一维控制分解 $E \oplus F \oplus G$, 且包含指标分别为 $\dim(E)$ 和 $\dim(E)+1$ 的周期轨,那么此同宿类支撑一个沿子丛 F 方向 Lyapunov 指数退化的非双曲测度, 且此测度的支集为整个同宿类.

我们在第 5.4 节给出了 [22] 的一个更一般的结果：通有情况下,如果一个同宿类包含不同指标周期轨,那么其支撑一个支集为整个同宿类的非双曲遍历测度. 这个结果同样也是定理 2 的一个推广.

持续存在性 人们同样也想考虑非双曲遍历测度会不会持续存在,比如猜测 2. 文献 [12] 对有多个退化 Lyapunov 指数的遍历测度提出一个类似的问题：

问题 3 *是否存在微分同胚空间中的一个开集 $\mathcal{U}$,使得任何 $f \in \mathcal{U}$ 都有一个多个 Lyapunov 指数退化的遍历测度?*

其他问题 我们可以提出关于同宿类支撑的非双曲遍历测度更多的问题,下面是 Bonatti 提出的两个：

(1) *是否存在这样的非双曲遍历测度其测度熵为正?*

文献 [13] 对于包含不同指标周期轨的同宿类给出这个问题的一个解答.

(2) *考虑所有第 i 个 Lyapunov 指数退化的遍历测度集合的闭包,这个集合是不是一个凸集?*

1.2.4 Dominated splitting over Lyapunov stable aperiodic classes (Chinese)

人们也会研究一类有压缩邻域的特殊链回复类,比如文献 [7, 19, 37, 62, 80]. 称一个紧致不变集合 K 是 **Lyapunov 稳定**的,如果对于 K 的任何邻域 U,存在 K 的另外一个邻域 V, 使得对于 $n \geq 0$,我们有 $f^n(V) \subset U$. 如果 K 对于 f 和 f^{-1} 都是 Lyapunov 稳定,我们称 K 是**双向 Lyapunov 稳定**.

由 [34, 39, 81],我们知道,对于远离同宿扰动（或者只是同宿切）的 C^1 通有微分同胚,一个非周期类是中心一维部分双曲的. 文献 [37, 80] 证明了远

离同宿扰动的 C^1 通有微分同胚,任何非周期类都不是 Lyapunov 稳定的. 而 [19] 的结果表明,存在 $\mathrm{Diff}^1(M)$ 的一个开子集 $\mathcal{U}$, 满足对于$\mathcal{U}$ 中通有的微分同胚 f 包含无穷多个双向 Lyapunov 稳定的没有非平凡控制分解的非周期类. 在文献 [62] 中,Potrie 证明了对于 C^1 通有微分同胚,如果一个同宿类是双向 Lyapunov 稳定的,那么其上必有非平凡的控制分解, 而且在某些情况下,这个同宿类是整个流形. 关于 Lyapunov 稳定的同宿类的其他一些结果可见 [62, 7].

我们陈述 Crovisier 在2015年国际数学家大会（ICM）报告中提出的关于非周期类的一个猜测,这个猜测表明 C^1 通有远离同宿扰动(或同宿切)的微分同胚是没有非周期类的.

猜测 5 ([36]) *考虑一个 C^1 通有微分同胚 $f \in \mathrm{Diff}^1(M)$ 以及 f 的一个非周期类 Λ.假设 $E^s \oplus E^c \oplus E^u$ 是 Λ 上的控制分解, 满足 E^s 是最大的压缩子丛而 E^u 是最大的扩张子丛.那么子丛 E^c 维数至少是二维,并且 E^c 没有非平凡的控制分解.*

下面的定理是本文的第三个结果,其断言对于 C^1 通有微分同胚,一个 Lyapunov 稳定的非周期类上的非平凡控制分解实际上是一个部分双曲分解. 这给出了猜测 5 的一个部分解答.

定理 C. *对通有的微分同胚 $f \in \mathrm{Diff}^1(M)$,如果一个 Lyapunov 稳定的非周期类 Λ 有控制分解 $T_\Lambda M = E \oplus F$, 那么或者子丛 E 是压缩的,或者子丛 F 是扩张的.*

定理 C 的一个推论表明,如果我们考虑一个 Lyapunov 稳定的非周期类上的控制分解 $E^s \oplus E^c \oplus E^u$, 满足 E^s 是最大的压缩子丛而且 E^u 是最大的扩张子丛,那么子丛 E^c 没有任何非平凡控制分解. 另外,按照 [36, 37] 中关于对 [33] 中构造的“中心模型”的讨论,我们可以知道此时 E^c 至少是二维. 这样我们知道猜测 5 对于 Lyapunov 稳定的非周期类是成立的.

这里我们指出定理 C 对于同宿类是不成立的,这表明伪周期假设是必要的. 文献 [27] 构造了 $\mathbb{T}^4$ 上一个持续传递的微分同胚,从而整个流形是双向 Lyapunov 稳定的同宿类. 而这个同宿类只有一个非平凡控制分

解 $E \oplus F$,其中 $\dim(E) = \dim(F) = 2$. 另外,流形上含有各种可能指标的周期鞍点,从而子丛 E 不压缩,子丛 F 不扩张.

由定理 C,我们可以得到子丛 E 和 F 中的一个（实际上只有一个）是双曲的,但是我们并不知道双曲的是哪个从. 对于 Lyapunov 稳定的链回复类中的任何点 x,其不稳定集 $W^u(x) = \{y \in M : \lim_{n\to+\infty} d(f^{-n}(x), f^{-n}(y)) = 0\}$ 也包含在这个类里. 从而如果子丛 F 是扩张的,由 [51],这个 Lyapunov 稳定的非周期类就是由切于 F 方向的不稳定流形片构成的,进而可以得到其不是极小的. 我们猜测这种现象是不会发生的.

猜测 6 对于通有的微分同胚 $f \in \mathrm{Diff}^1(M)$,如果一个 Lyapunov 稳定的非周期类 Λ 有控制分解 $T_\Lambda M = E \oplus F$,那么子丛 E 是压缩的.

Bonatti 和 Shinohara有一个研究课题就是构造有非平凡控制分解的非周期类. 由于非周期类不是孤立的,这种例子并不容易构造. 即使是非孤立的 Lyapunov 稳定的同宿类的构造也是比较复杂的,见 [26].

实际上,文献 [21] 中构造的双向 Lyapunov 稳定的非周期类都是极小的康托集,而且其上都没有非平凡控制分解. 基于此事实,我们有下面的猜测.易见如果猜测 6 成立,那么下面的猜测也是成立的.

猜测 7 对于通有的微分同胚 $f \in \mathrm{Diff}^1(M)$,一个双向 Lyapunov 稳定的非周期类上没有任何非平凡控制分解.

文献 [82] 中的结果是与猜测 7 相关的. 其结果是,如果一个微分同胚是极小的（从而整个流形是一个双向 Lyapunov 稳定的非周期类）,那么这个同胚没有任何非平凡控制分解.

1.3 Introduction (English)

1.3.1 Backgrounds

One of the goals of dynamical systems is to describe most of the systems, i.e. a dense, residual, or open and dense subset. In the 1960's, Abraham and Smale [5, 68] found that the set of *hyperbolic diffeomorphisms* (i.e. satisfying the axiom A and the no-cycle condition) is not dense in the space of dynamical systems. Hence the study of dynamics beyond uniform hyperbolicity became a major problem for dynamists since then. The lack of hyperbolicity may be characterized in different ways. Let M be a compact smooth manifold without boundary of dimension d and denote by $\mathrm{Diff}^r(M)$ the space of C^r-diffeomorphisms of M for any $r \geq 1$.

– Thanks to the contributions of Liao, Mañé, Aoki and Hayashi [53, 54, 6, 50] to the C^1-stability conjecture, it is known that any non-hyperbolic diffeomorphism can be perturbed as a diffeomorphism with a non-hyperbolic periodic orbit. In particular a C^1-generic diffeomorphism (i.e. diffeomorphism in a dense G_δ subset of $\mathrm{Diff}^1(M)$) which is not hyperbolic has arbitrarily "weak" periodic orbits.

– Pesin's theory [60] weakens the notion of hyperbolicity (non-uniform hyperbolicity), and gives a possible approach to characterize non-hyperbolic behavior through the invariant measures.

– Another global obstruction to hyperbolicity is the famous homoclinic bifurcation in the spirit of Poincaré: homoclinic tangency or heterodimensional cycle, which are still related to periodic orbits. Palis conjectured [58, 59] that any diffeomorphism can be C^r-approximated by one which is hyperbolic or by one which exhibits a homoclinic bifurcation. These bifurcations have strong dynamical consequences and can sometimes be strengthened as robust obstructions to hyperbolicity, see [20]. Several progresses have been obtained in the direction of this conjecture [24, 33, 34, 37, 39, 64].

The above discussions are about the global obstructions of hyperbolicity. One also would like to study the "local" non-hyperbolicity, that is the non-hyperbolicity on a basic set of the dynamics. The dynamics concentrates essentially on the set of points that have some recurrence properties, for example the *chain recurrent set*, which splits into disjoint indecomposable *chain recurrence classes*, see [30]. For C^1-generic diffeomorphisms, periodic orbits are dense in the chain recurrent set and any chain recurrence class containing a periodic point p coincides with its *homoclinic class*: the closure of transverse intersections between the stable and unstable manifolds of $\mathrm{Orb}(p)$, see [15, 32].

Hence two homoclinic classes either coincide or are disjoint for C^1-generic diffeomorphisms, but this is not true in general. Also, homoclinic classes are generally infinite in number, even for generic diffeomorphisms. A chain recurrence class without any periodic point is called an *aperiodic class*.

This thesis will characterize the non-hyperbolicity of a homoclinic class via weak periodic orbits inside or via non-hyperbolic ergodic measures supported on it. We also state a result for aperiodic classes and do some discussions related to Palis' conjecture.

1.3.2 Hyperbolicity versus weak periodic orbits

We study a problem that is in spirit of the works of the stability conjecture but with an "intrinsic" nature. The stability conjecture formulated by Palis and Smale claims that if a diffeomorphism f is structurally stable then it is hyperbolic (i.e. satisfying the axiom A and the no-cycle condition). A stronger version of the conjecture claims that if f is Ω-stable then it is hyperbolic. These two remarkable conjectures are solved by Mañé [55] and Palis [57], respectively.

During the long way of study of the stability conjectures, the attention was more and more concentrated on periodic orbits of the (unperturbed) diffeomorphism f as well as its perturbations g. Liao [53] and Mañé [54] raised independently a conjecture (more precisely, a problem without a tentative answer), known as the star conjecture, stating that if f has no, robustly, non-hyperbolic periodic orbits then it is hyperbolic. Being an assumption, the star condition is clearly weaker than the Ω-stability. Hence the star conjecture is regarded as another (strong) version of the stability conjecture. It has been solved by Aoki and Hayashi [6, 50]. To compare more precisely with our result below we state their results in a generic version.

Recall that, By Oseledets' Theorem [56], for an invariant ergodic measure μ of a diffeomorphism $f\in \mathrm{Diff}^1(M)$, there are d numbers $\chi_1(\mu,f)\leq\chi_2(\mu,f)\leq\cdots\leq\chi_d(\mu,f)$, such that, for μ-a.e. point $x\in M$, and for any $v\in T_xM\setminus\{0\}$, we have $\lim_{n\to+\infty}\frac{1}{n}\log\|Df^n(v)\|=\chi_i$ for some $i\in\{1,2,\cdots,d\}$. The d numbers $\chi_i(\mu,f)$ are called the *Lyapunov exponents* of the measure μ. We call μ a *hyperbolic* measure, if all its Lyapunov exponents are non-zero. In particular, if μ is an atomic measure distributed averagely on a periodic orbit $\mathrm{Orb}(p)$, the Lyapunov exponents of μ are also called the *Lyapunov exponents* of the periodic orbit $\mathrm{Orb}(p)$ (or of the periodic point p).

Theorem 1 ([6, 50])**.** *For generic $f\in\mathrm{Diff}^1(M)$, if f is not hyperbolic, then there are weak periodic orbits: there exists a sequence of periodic orbits that have a Lyapunov exponent converging to 0.*

One would consider naturally whether Theorem 1 is true or not for the basic sets of a diffeomorphism, that is for homoclinic classes. Recall that two hyperbolic periodic points p and q are *homoclinically related*, if $W^u(\mathrm{Orb}(p))$

has non-empty transverse intersections with $W^s(\mathrm{Orb}(q))$ and vice versa. A homoclinic class of a hyperbolic periodic point p also equals the closure of the set of periodic points that are homoclinically related to it. Below is a problem raised in [39].

Problem (Problem 1.8 of [39])**.** *Is it true that for generic $f \in \mathrm{Diff}^1(M)$, and any homoclinic class $H(p)$, either*

- *$H(p)$ is hyperbolic, or*
- *$H(p)$ contains weak periodic orbits homoclinically related to p ?*

In fact, it is straightforward to prove (following the classical proof of the stability conjecture) that, there must be a weak periodic orbit arbitrarily near $H(p)$. But one do not know whether the weak periodic orbits are contained inside. Of course, if the homoclinic class $H(p)$ is assumed to be isolated, then being "near" will be equivalent to being "inside". The point is that here $H(p)$ is not known to be isolated (there may be infinitely many classes), even for C^1-generic diffeomorphisms. Now we state the first result of the thesis, which gives a positive answer to this problem, see [69, 71].

Theorem A. *For generic $f \in \mathrm{Diff}^1(M)$, if a homoclinic class $H(p)$ of f is not hyperbolic, then $H(p)$ contains weak periodic orbits homoclinically related to p.*

Notice that, in contrast to the classical arguments of the stability conjecture, here Theorem A claims there must be a weak periodic orbit not only near, but actually inside $H(p)$. This is the main point of Theorem A. At each step of the proof, the periodic orbits created by perturbations have to be guaranteed to lie strictly inside the homoclinic class. In this sense we say the problem is of an "intrinsic" nature, and the classical arguments of the stability conjecture do not pass through.

In [55], Mañé introduced a very useful lemma (Theorem **II.1**) to get weak periodic orbits under certain hypothesis. The statement is very technical and the original proof of Mañé is difficult, thus we will not state it here. Based on a modification of Mañé's proof, Bonatti, Gan and Yang have a result for homoclinic classes, see [25]. We point out here that, different from Theorem **II.1** of [55] and the result of [25], there is a genericity assumption in Theorem A. That is to say, the conclusion of Theorem A is a perturbation result and may not be valid for all diffeomorphisms.

We point out here that the genericity assumption is essential in Theorem A. There is an example given by [66, 28], showing that, by destroying hyperbolic horseshoes in a parameterized family of diffeomorphisms on surface, there is a diffeomorphism which is on the boundary of the set of hyperbolic diffeomorphisms, and which has a homoclinic class containing a homoclinic tangency inside. Hence the homoclinic class is not hyperbolic because of the existence of the tangency. Moreover, all the Lyapunov exponents of all ergodic measures are uniformly bounded away from 0, and in particular all the Lyapunov exponents of all periodic orbits contained in the class are uniformly bounded away form 0.

Recall that a *dominated splitting* $E \oplus F$ on an invariant compact set Λ is an Df-invariant splitting of $T_\Lambda M$ and the norm of Df along E is controlled by that along F, and Λ is *partially hyperbolic* if $T_\Lambda M$ splits into three bundles $E^s \oplus E^c \oplus E^u$ which is a dominated splitting such that the extremal bundles are hyperbolic and the center bundle is neutral (see Definition 2.6). Examples of non-hyperbolic homoclinic classes with a dominated splitting can be found in like [31, 40, 65], but the homoclinic classes in these examples contain weak periodic orbits. For C^2 diffeomorphisms on surfaces, by the conclusions of [65], one can not give a non-hyperbolic homoclinic class with domination and without weak periodic orbits, which is unknown in the C^1 dynamics. Hence we have the following question.

Question 1. *Does there exist a non-hyperbolic homoclinic class $H(p)$ with a non-trivial dominated splitting for a diffeomorphism $f \in \mathrm{Diff}^1(M)$ satisfying that all the Lyapunov exponents of all periodic orbits homoclinically related to* $\mathrm{Orb}(p)$ *are uniformly bounded away from 0 ?*

There are other conjectures aimed to give a dichotomy of global dynamics. Recall that a *homoclinic tangency* of a hyperbolic periodic point p is a non-transverse intersection between $W^u(p)$ and $W^s(p)$. A diffeomorphism exhibits a *heterodimensional cycle* if there are two hyperbolic periodic points p and q with different *indices* (i.e. stable dimensions) such that $W^s(p) \cap W^u(q) \neq \emptyset$ and $W^s(q) \cap W^u(p) \neq \emptyset$. It is obvious that any diffeomorphism with either a tangency or a heterodimensional cycle is not hyperbolic. Palis [58, 59] conjectured that these two phenomenons are the only obstacles for hyperbolicity. This conjecture was solved by [64] in dimension 2, and for higher dimension, works by [34, 39, 37] got progress: generically, far from homoclinic bifurcations, the systems has some weak hyperbolicity (partially hyperbolic or essentially hyperbolic). Based on the results afterwards, Bonatti and Díaz conjectured that the union of diffeomorphisms that are hyperbolic and those with heterodimensional cycles is dense in the space of diffeomorphisms, see [14, 20].

By the Franks' lemma [42, 48], we can perturb weak periodic orbits to get periodic orbits with different indices. But it is not clear whether these periodic orbits are still contained in the non-hyperbolic homoclinic class after perturbation. Thus we have the following conjecture, which is an intrinsic version of Palis conjecture for homoclinic classes.

Conjecture 1 ([14, 17, 20, 34])**.** *For generic $f \in \mathrm{Diff}^1(M)$, a non-hyperbolic homoclinic class $H(p)$ contains a periodic point $q \in H(p)$, whose index is different from that of p.*

1.3.3 Hyperbolicity versus non-hyperbolic measures

Clearly if a diffeomorphism has a non-hyperbolic ergodic measure it can not be hyperbolic (i.e. satisfying Axiom A and the no cycle condition). The converse is not true in general, see [10, 28], but we may expect that it is the case for typical systems, as conjectured by Díaz and Gorodetski in [41].

Conjecture 2 (Conjecture 1 of [41]). *There is an open dense subset $\mathcal{U} \subset \mathrm{Diff}^r(M)$ where $r \geq 1$, such that, every diffeomorphism $f \in \mathcal{U}$ either is uniformly hyperbolic or has an ergodic non-hyperbolic invariant measure.*

Notice that Conjecture 2 is true if we replace *open dense* by just *dense*, which is a direct corollary of the star conjecture. An alternative (weaker) slightly different reformulation is to consider generic diffeomorphisms. As a corollary of Theorem 2 below, the generic version of Conjecture 2 holds for *tame* diffeomorphisms: those such that all homoclinic classes are robustly isolated, see [41].

Theorem 2 (Theorem 1 of [41]). *For generic $f \in \mathrm{Diff}^1(M)$, if a homoclinic class contains periodic points with different indices, then it supports a non-hyperbolic ergodic measure μ.*

As we have mentioned in Section 1.3.2, in general, a homoclinic class is not isolated, even for C^1-generic diffeomorphisms. Theorem 2 shows that to settle the generic version of Conjecture 2, it is enough to prove that every *wild* diffeomorphism (one which is not tame) has a homoclinic class which contains periodic points with different indices.

One also expects that the generalization of Theorem 2 holds, as conjectured in [41].

Conjecture 3 (Conjecture 2 of [41]). *For generic $f \in \mathrm{Diff}^1(M)$, every homoclinic class either is uniformly hyperbolic or supports an ergodic non-hyperbolic invariant measure.*

We point out here that, Conjecture 3 is not true for a general diffeomorphism. The example of [66, 28] is a homoclinic class which is not hyperbolic because of the existence of homoclinic tangencies, but it is uniformly hyperbolic in the measure sense. To be precise, all Lyapunov exponents of all invariant measures supported on the homoclinic class are uniformly bounded away from 0. However, a homoclinic tangency is not persistent under perturbations, hence one expects a positive answer to Conjecture 3, which is the next result of this thesis. It is a joint work with C. Cheng, S. Crovisier, S. Gan and D. Yang, see [29].

Theorem B. *For generic $f \in \mathrm{Diff}^1(M)$, a homoclinic class $H(p)$ which is not hyperbolic supports a non-hyperbolic ergodic measure μ.*

Let us emphasize that the same result does not hold for aperiodic classes: [18] builds an open set of diffeomorphisms whose C^1-generic elements have aperiodic classes which only support hyperbolic ergodic measures. We do not have an answer to Conjecture 2. A possible approach would be to answer the following problem, see [2, Conjecture 1], [14, Conjecture 1] or [36, Section 6.2.Ia].

Conjecture 4. *Any C^1-generic diffeomorphism which is not hyperbolic has a non-hyperbolic homoclinic class.*

Let us comment the technics for getting ergodic measures with one zero Lyapunov exponent.

In the special case a homoclinic class $H(p)$ has a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$, where $\dim(E^c) = 1$, then the class is hyperbolic if and only if all its ergodic measures have a non-zero center Lyapunov exponent, with the same sign; any hyperbolic measure is approximated by hyperbolic periodic orbits in the class with the same index (see [34] and Proposition 2.45 below). Hence if $H(p)$ is not hyperbolic, either it contains a non-hyperbolic ergodic measure (as required), or it contains two hyperbolic periodic orbits with different indices. In this second case, it is possible to build a non-hyperbolic measure by mixing the two period orbits, using certain shadowing properties. This has been developed by many works [46, 41, 22, 13] among others. The partial hyperbolicity on a subset of the class is enough for this argument, for example Theorem 2.

The main difficulty in the proof of Theorem B is to obtain periodic points with different indices in a same non-hyperbolic homoclinic class. This is achieved under new settings, thanks to Theorem A (which produces weak periodic orbits) combined to [17] (in order to turn them to heterodimensional cycles). There are still cases where we do not manage to get such periodic points (see the discussion in Section 4.5 below), but the existence of non-hyperbolic measures is ensured then.

Let us discuss more properties of these non-hyperbolic measures.

Several vanishing Lyapunov exponents The non-hyperbolic ergodic measures mentioned above can only be assured to have one vanishing Lyapunov exponent. The example in [12] shows that there exist iterated function systems (IFS) persistently exhibiting non-hyperbolic ergodic measures with all the Lyapunov exponents equal to zero. For homoclinic classes, one can ask the following question:

Question 2. *Under what kind of assumption, dose there exist a non-hyperbolic ergodic measure supported on a homoclinic class with more than one vanishing Lyapunov exponents ?*

Results of [41] tell that, generically, if a homoclinic class contains periodic points of different indices, then the class supports a non-hyperbolic ergodic measure, see Theorem 2. Inspired by this result, we would like to consider the question that:
If the homoclinic class contains periodic points of indices i and j respectively, where $i < j$, does there exist an ergodic measure supported on the homoclinic class such that all its $(i+1)^{th}$ to j^{th} Lyapunov exponents vanish ?

Obviously, it is not true if the homoclinic class admits a dominated splitting $E \oplus F$ with $i < \dim(E) < j$. What happens when there is no such dominated splitting over the class? We state a recent joint work with J. Zhang [72], which partially answers Question 2. Recall that for a dominated splitting $E \oplus F$, we call $\dim(E)$ its *index*.

Theorem 3 ([72]). *For generic $f \in \mathrm{Diff}^1(M)$, consider a hyperbolic periodic point p. Assume the homoclinic class $H(p)$ satisfies the following properties:*

– *there exist hyperbolic periodic points q_1, q_2 contained in $H(p)$ of indices i and j respectively, where $i < j$,*

– *there is no dominated splitting of index k for any $k = i+1, \cdots j-1$.*

Then there exists an ergodic measure μ whose support is $H(p)$, such that the k^{th} Lyapunov exponents of μ vanish, for all $k = i+1, \cdots j$.

We would not give the proof of Theorem 3 in the thesis.

Support Theorem B and Theorem 2 can obtain non-hyperbolic ergodic measures supported on the homoclinic class. One consider a question that:
Can the supports of the non-hyperbolic ergodic measures be the whole homoclinic class ?

It is proved in some cases, for example Theorem 3 above. A previous result by [22] proved that, for C^1-generic diffeomorphisms, if a homoclinic class admits a dominated splitting into three bundles $E \oplus F \oplus G$ with $\dim(F) = 1$ and if it contains both hyperbolic periodic orbits of indices $\dim(E)$ and $\dim(E)+1$, then there is a non-hyperbolic ergodic measure μ whose Lyapunov exponent along the bundle F vanishes and whose support is the whole homoclinic class.

We give in Section 5.4 a general result of [22], which tells that, generically, if a homoclinic class contains periodic points of different indices, then it supports a non-hyperbolic ergodic measure with full support. This also generalized Theorem 2.

Robustness of existence One also would like to see, whether the existence of non-hyperbolic ergodic measures are robust, as one can see in Conjecture 2. A similar question was raised in [12] for ergodic measures with more than one vanishing Lyapunov exponents.

Question 3. *Does there exist an open set $\mathcal{U}$ of diffeomorphisms such that any $f \in \mathcal{U}$ has an ergodic measure with more than one vanishing Lyapunov exponents ?*

Further questions Further questions about the non-hyperbolic ergodic measures supported on the class may be asked. Bonatti proposed the following two:
(1) *Is there such a measure with positive entropy ?*
This has been obtained in [13] for homoclinic classes containing periodic points with different indices.
(2) *Consider the closure of the set of ergodic measures whose i^{th} Lyapunov exponent vanishes. Is it a convex set ?*

1.3.4 Dominated splitting over Lyapunov stable aperiodic classes

One also would like to study some special chain recurrence class that admits trapping neighborhoods, for example [7, 19, 37, 62, 80]. Recall that an invariant compact set K is called *Lyapunov stable*, if for any neighborhood U of K, there is another neighborhood V of K, such that $f^n(V) \subset U$ for all $n \geq 0$. And K is called *bi-Lyapunov stable*, if it is Lyapunov stable both for f and f^{-1}.

By [34, 39, 81], for C^1-generic diffeomorphisms far from homoclinic bifurcations (or just homoclinic tangencies), an aperiodic class is partially hyperbolic with center bundle of dimension 1. The papers [37, 80] prove that any aperiodic class of a C^1-generic diffeomorphism that is far from homoclinic bifurcations can not be Lyapunov stable. But in [19], it is showed that there is an open set $\mathcal{U} \subset \mathrm{Diff}^1(M)$, such that for generic $f \in \mathcal{U}$, there are infinitely many bi-Lyapunov stable aperiodic classes which admits no dominated splitting. In [62], Potrie proved that for C^1-generic diffeomorphisms, if a homoclinic class is bi-Lyapunov stable, then it admits a non-trivial dominated splitting, and under some more hypothesis, it is the whole manifold. There are also many other results for Lyapunov stable homoclinic classes in [62, 7].

We state here a conjecture for aperiodic classes by Crovisier in his report to ICM 2015, which implies the non-existence of aperiodic classes for C^1-generic diffeomorphisms far from homoclinic bifurcations (or homoclinic tangencies).

Conjecture 5 ([36])**.** *Let $f \in \mathrm{Diff}^1(M)$ be a C^1-generic diffeomorphism and Λ be an aperiodic class of f. Assume that $E^s \oplus E^c \oplus E^u$ is the dominated splitting on Λ such that E^s (resp. E^u) is the maximal contracted (resp. expanded) sub-bundle, then E^c has dimension at least two and admits no non-trivial dominated splitting.*

The following theorem is the third result of the thesis, see [70]. It claims that a non-trivial dominated splitting on a Lyapunov stable aperiodic class is actually a partially hyperbolic splitting for C^1-generic diffeomorphisms. This gives a partial answer to Conjecture 5.

Theorem C. *For generic $f \in \mathrm{Diff}^1(M)$, if a Lyapunov stable aperiodic class Λ of f admits a dominated splitting $T_\Lambda M = E \oplus F$, then one (and only one) of the two cases happens: E is contracted or F is expanded.*

As a consequence of Theorem C, for C^1-generic diffeomorphisms, if we consider the dominated splitting $E^s \oplus E^c \oplus E^u$ on a Lyapunov stable aperiodic class, such that E^s (resp. E^u) is the maximal contracted (resp. expanded) sub-bundle, then the sub-bundle E^c admits no non-trivial dominated splitting. Moreover, with the arguments of [36, 37], using the "central model" constructed in [33], one knows that the dimension of E^c is at least two. Hence Conjecture 5 holds for Lyapunov stable aperiodic classes.

We point out that the statement of Theorem C fails for homoclinic classes, which shows that the aperiodicity is an essential assumption. The paper [27] constructs a robustly transitive diffeomorphism of $\mathbb{T}^4$, hence the whole manifold is a bi-Lyapunov stable homoclinic class, and it admits only one dominated splitting $E \oplus F$ with $\dim(E) = \dim(F) = 2$. Moreover, there are periodic saddles of all possible indices, which implies that neither E is contracted nor F is expanded.

By Theorem C, we can get that one (and only one actually) of the two bundles E and F is hyperbolic, but we do not know which one it is. For any point x contained in a Lyapunov stable chain recurrence class, the unstable set $W^u(x) = \{y \in M : \lim_{n\to+\infty} d(f^{-n}(x), f^{-n}(y)) = 0\}$ is also contained in the class. Hence if the bundle F is expanded, by [51], the Lyapunov stable aperiodic class is foliated by unstable manifolds that are tangent to F, and thus it can not be minimal. We conjecture that such phenomenon can not happen.

Conjecture 6. *For generic $f \in \mathrm{Diff}^1(M)$, if a Lyapunov stable aperiodic class Λ of f admits a dominated splitting $T_\Lambda M = E \oplus F$, then the bundle E is contracted.*

Bonatti and Shinohara have a programme to construct aperiodic classes with a non-trivial dominated splitting. Since aperiodic classes are not isolated, such examples are not easy to construct. Even the construction of non-isolated Lyapunov stable homoclinic classes is delicate, see [26].

Actually, each of the bi-Lyapunov stable aperiodic classes constructed in [21] is a minimal Cantor set, and admits no non-trivial dominated splitting. Hence we have the second conjecture that is obviously true if Conjecture 6 is true.

Conjecture 7. *For generic $f \in \mathrm{Diff}^1(M)$, if an aperiodic class Λ of f is bi-Lyapunov stable, then it admits no non-trivial dominated splitting.*

There is a result [82] related to Conjecture 7 stating that, if a diffeomorphism is minimal (and hence the whole manifold is a bi-Lyapunov stable aperiodic class), then it admits no dominated splitting.

Organization of the thesis In Chapter 2, we give the definitions and some known results. Chapter 3 proves four multi-connecting propositions that we need to apply in the proofs of the main theorems. We present the proof of Theorem A, and give some applications in Chapter 4. The proofs of Theorem B and Theorem C are given in Chapter 5 and Chapter 6 respectively.

Chapter 2

Preliminaries

In this chapter, we give some definitions and some well known results. Denote by $\mathrm{Diff}^1(M)$ the space of C^1-diffeomorphisms of M, where M is a compact smooth manifold without boundary and $\dim(M) = d \geq 2$.

2.1 Hyperbolicity

Definition 2.1. Consider a diffeomorphism $f \in \mathrm{Diff}^1(M)$. Assume K is an invariant compact set and E is a Df-invariant sub-bundle of $T_K M$. For any two constants $C > 0$ and $\lambda \in (0,1)$, we say that the bundle E is (C,λ)-*contracted* if

$$\|Df^n|_{E(x)}\| < C\lambda^n,$$

for all $x \in K$ and all $n \geq 1$. We say that E is (C,λ)-*expanded* if it is (C,λ)-*contracted* with respect to f^{-1}. If the tangent bundle of Λ has an invariant splitting $T_K M = E^s \oplus E^u$, such that, E^s is (C,λ)-contracted and E^u is (C,λ)-expanded for some constants $C > 0$ and $\lambda \in (0,1)$, then we call K a *hyperbolic set* and $\dim(E^s)$ the *index* of the hyperbolic splitting. Moreover, if a periodic orbit $\mathrm{Orb}(p)$ is a hyperbolic set, then we call p a hyperbolic periodic point, and the dimension of the contracted bundle E^s in the hyperbolic splitting is called the *index* of p, denoted by $\mathrm{Ind}(p)$.

Definition 2.2. For any point $x \in M$ and any number $\delta > 0$, we define the *local stable set* and *local unstable set* of x of size δ respectively as follows:

$$W^s_\delta(x) = \{y : \forall n \geq 0, d(f^n(x), f^n(y)) \leq \delta;\ \text{and} \lim_{n\to+\infty} d(f^n(x), f^n(y)) = 0\};$$

$$W^u_\delta(x) = \{y : \forall n \geq 0, d(f^{-n}(x), f^{-n}(y)) \leq \delta;\ \text{and} \lim_{n\to+\infty} d(f^{-n}(x), f^{-n}(y)) = 0\}.$$

We define the *stable set* and *unstable set* of x respectively as follows:

$$W^s(x) = \{y : \lim_{n\to+\infty} d(f^n(x), f^n(y)) = 0\};$$

$$W^u(x) = \{y : \lim_{n\to+\infty} d(f^{-n}(x), f^{-n}(y)) = 0\}.$$

Remark 2.3. (1) *It is obvious that, for any $\delta > 0$, we have*

$$W^s(x) = \cup_{n\geq 0} f^{-n}(W^s_\delta(f^n(x)))$$

and

$$W^u(x) = \cup_{n\geq 0} f^{n}(W^u_\delta(f^{-n}(x))).$$

(2) *To belong to a same stable set is an equivalent relation, thus two stable sets either coincide or are disjoint with each other. Similarly with the unstable set.*

For hyperbolic sets, the (local) stable (resp. unstable) set has the following properties, see for example [51].

Lemma 2.4. *If K is a hyperbolic set and $T_K M = E^s \oplus E^u$ is the hyperbolic splitting, then there is a number $\delta > 0$, such that, for any $x \in K$, the local stable (resp. unstable) set $W^s_\delta(x)$ (resp. $W^u_\delta(x)$) is an embedding disk with dimension $dim(E^s)$ (resp. $dim(E^u)$) and is tangent to E^s (resp. E^u) at x. Moreover, the stable (resp. unstable) set $W^s(x)$ (resp. $W^u(x)$) of x is an immersed submanifold of M.*

Definition 2.5. Assume that f is a diffeomorphism in $\mathrm{Diff}^1(M)$ and $p, q \in M$ are two hyperbolic periodic point of f, we say p and q are *homoclinically related*, if $W^u(\mathrm{Orb}(p))$ has non-empty transverse intersections with $W^s(\mathrm{Orb}(q))$, and $W^s(\mathrm{Orb}(p))$ has non-empty transverse intersections with $W^u(\mathrm{Orb}(q))$, denoted by $W^u(\mathrm{Orb}(p)) \pitchfork W^s(\mathrm{Orb}(q)) \neq \emptyset$ and $W^s(\mathrm{Orb}(p)) \pitchfork W^u(\mathrm{Orb}(q)) \neq \emptyset$. We call the closure of the set of periodic orbits homoclinically related to $\mathrm{Orb}(p)$ the *homoclinic class* of p and denote it by $H(p, f)$ or $H(p)$ for simplicity.

Definition 2.6. Consider a diffeomorphism $f \in \mathrm{Diff}^1(M)$, an integer $m > 0$ and a constant $\lambda \in (0, 1)$. An invariant compact set K is said to have an (m, λ)-*dominated splitting*, if the tangent bundle has an Df-invariant splitting $T_K M = E \oplus F$ such that

$$\|Df^m|_{E(x)}\| \cdot \|Df^{-m}|_{F(f^m x)}\| < \lambda,$$

for any $x \in K$. We call $\dim(E)$ the *index* of the dominated splitting. We say K has a *partially hyperbolic splitting*, if the tangent bundle has an Df-invariant splitting $T_K M = E^s \oplus E^c \oplus E^u$, which are both dominated splittings considering $(E^s \oplus E^c) \oplus E^u$ and $E^s \oplus (E^c \oplus E^u)$ satisfying that, the extreme bundles are hyperbolic: the bundle E^s is contracted and the bundle E^u is expanded, the central bundle E^c is neither contracted nor expanded, and moreover, at least one of E^s and E^u is non-trivial.

We give another definition of dominated splitting by using one parameter. One can see that the two definitions are equivalent.

Definition 2.7. Assume $f \in \mathrm{Diff}^1(M)$ and T is a positive integer. A T-*dominated splitting* over an invariant compact set K is a continuous Df-invariant splitting $T_K M = E \oplus F$, such that,

$$\|Df^T|_{E(x)}\| \cdot \|Df^{-T}|_{F(f^T(x))}\| < \tfrac{1}{2},$$

for any $x \in K$. We say K has a *dominated splitting*, if K has a T-dominated splitting for some T. Moreover, the dimension of the bundle E is called the *index* of the dominated splitting.

Remark 2.8. (1) *We point out here that if an invariant compact set K has two dominated splittings $T_K M = E_1 \oplus F_1 = E_2 \oplus F_2$ such that $dim(E_1) \leq dim(E_2)$, then we have $E_1 \subset E_2$. Hence two dominated splittings on an invariant compact set with the same index would coincide.*

(2) *A T-dominated splitting (or (m, λ)-dominated splitting) over an invariant compact set K of f can be extended to the maximal invariant compact set of a neighborhood U of K for any diffeomorphism g in a neighborhood $\mathcal{U}$ of f.*

By [47], there is always an *adapted metric* for a dominated splitting, that is to say, an (m, λ)-dominated splitting is a $(1, \lambda)$-dominated splitting by considering a metric equivalent to the original one. Also, it is obvious that an (m, λ)-dominated splitting is always an (mN, λ)-dominated splitting for any positive integer N.

2.2 Lyapunov exponents

For an f-invariant measure μ, we list all its Lyapunov exponents as $\chi_1(\mu, f) \leq \chi_2(\mu, f) \leq \cdots \leq \chi_d(\mu, f)$. Denote by $\chi_i(\mu)$ if there is no ambiguity. We define a function

$$L_i(\mu, f) = \liminf_{m \to +\infty} \mathrm{Int}\, L_i^{(m)}(x, f) d\mu(x),$$

where

$$L_i^{(m)}(x, f) = \frac{1}{m} \log \| \wedge^i Df^m(x)\|.$$

Then $\chi_i(\mu, f) = L_{d-i+1}(\mu, f) - L_{d-i}(\mu, f)$. In particular, if μ is ergodic, then for μ-a.e. $x \in M$, we have

$$L_i(\mu, f) = \lim_{m \to +\infty} L_i^{(m)}(x, f),$$

and

$$\chi_i(\mu, f) = \lim_{m \to +\infty} (L_{d-i+1}^{(m)}(x, f) - L_{d-i}^{(m)}(x, f)).$$

For an atomic measure μ which is distributed averagely on a periodic orbit $\mathrm{Orb}(p)$, we denote by $\chi_1(p,f) \leq \chi_2(p,f) \leq \cdots \leq \chi_d(p,f)$ the Lyapunov exponents of μ, and also call them the *Lyapunov exponents* of the periodic point p. A hyperbolic saddle p of index i is said to be *center-dissipative* if $\chi_i(p,f) + \chi_{i+1}(p,f) < 0$.

2.3 Decomposition of dynamics

The dynamics concentrates essentially on the set of points which has some recurrence properties. In this section, we give some definitions of recurrence.

Definition 2.9. For a diffeomorphism $f \in \mathrm{Diff}^1(M)$ and a number $\varepsilon > 0$, we call a sequence of points $\{x_i\}_{i=a}^{b}$ of M an *ε-pseudo orbit of f*, if $d(f(x_i), x_{i+1}) < \varepsilon$ for any $i = a, a+1, \cdots, b-1$, where $-\infty \leq a < b \leq \infty$. An invariant compact set K is called a *chain transitive set*, if for any $\varepsilon > 0$, there is a periodic ε-pseudo-orbit contain in K and ε-dense in K.

Definition 2.10. Assume $f \in \mathrm{Diff}^1(M)$. We say a point y is *chain attainable* from x, if for any number $\varepsilon > 0$, there is an ε-pseudo orbit of f $(x_0, x_1, \cdots, x_n)$ such that $x_0 = x$ and $x_n = y$, and we denote it by $x \dashv y$. The *chain recurrent set* of a diffeomorphism $f \in \mathrm{Diff}^1(M)$, denoted by $R(f)$, is the union of the point x such that x is chain attainable from itself.

It is well known that the chain recurrent set $R(f)$ of f can be decomposed into a disjoint union of invariant compact "undecomposable" sets. More precisely, we give the definition as the following.

Definition 2.11. Assume $f \in \mathrm{Diff}^1(M)$. For any two points $x, y \in M$, denote by $x \vdash\dashv y$ if $x \dashv y$ and $y \dashv x$. Obviously $\vdash\dashv$ is an equivalent relation on $R(f)$, and an equivalent class of $\vdash\dashv$ is called a *chain recurrence class.*

Definition 2.12. Assume $f \in \mathrm{Diff}^1(M)$ and Λ is an invariant compact set of f. We say that Λ is *shadowable*, if for any $\varepsilon > 0$, there is $\delta > 0$, such that for any δ-pseudo orbit $\{x_i\}_{i=a}^{b} \subset \Lambda$ of f, where $-\infty \leq a < b \leq \infty$, there is a point $y \in M$, such that $d(f^i(y), x_i) < \varepsilon$ for all $a \leq i \leq b$.

Now we give another definition of a relation, denoted by $\prec$, that was first introduced in [8] and [44], also see [32].

Definition 2.13. Assume f is a diffeomorphism in $\mathrm{Diff}^1(M)$ and W is an open set of M. For any two points $x, y \in M$, we denote by $x \prec y$ if for any neighborhood U of x and any neighborhood V of y, there are a point $z \in M$ and an integer $n \geq 1$, such that $z \in U$ and $f^n(z) \in V$. We denote $x \prec_W y$ if for any neighborhood U of x and any neighborhood V of y, there is a piece of orbit $(z, f(z), \cdots, f^n(z))$ contained in W such that $z \in U$ and $f^n(z) \in V$.

Moreover, let K be a compact set of M, then we denote $x \prec K$ (resp. $x \prec_W K$) if there is a point $y \in K$, such that $x \prec y$ (resp. $x \prec_W y$). Similarly, one can define $x \dashv_W y$ and $x \dashv_K y$.

For the relation $\prec$, we have the following result, whose proof is similar to the proof of Lemma 6 in [32].

Lemma 2.14. *Assume that K is an invariant compact set. Then for any two neighborhoods $U_2 \subset U_1$ of K and any point $y \in U_1$ satisfying $y \prec_{U_1} K$, there is a point $y' \in U_2 \setminus K$, such that $y \prec_{U_1} y' \prec_{U_2} K$ and the positive orbit of y' is contained in U_2.*

It is obvious that $x \prec y$ implies $x \dashv y$, but the two relations are not equivalent. In [15], they have proved that for generic diffeomorphisms, the two relations are equivalent, see Theorem 2.46 below.

Definition 2.15. Assume $f \in \mathrm{Diff}^1(M)$. An invariant compact set $K \subset M$ is *Lyapunov stable for f*, if for any neighborhood U of K, there is another neighborhood V of K, such that $f^n(V) \subset U$ for all $n \geq 0$. We say that K is *bi-Lyapunov stable*, if K is both Lyapunov stable for f and for f^{-1}.

2.4 Pliss points and weak sets

Definition 2.16. Assume there is a dominated splitting $T_K M = E \oplus F$ over an invariant compact set K of a diffeomorphism $f \in \mathrm{Diff}^1(M)$ and $0 < \lambda < 1$. A point $x \in K$ is called an *(m,λ)-E-Pliss point* (resp. *(m,λ)-F-Pliss point*), if

$$\prod_{i=0}^{n-1} \|Df^m|_{E(f^{im}(x))}\| \leq \lambda^n \text{ (resp. } \prod_{i=0}^{n-1} \|Df^{-m}|_{F(f^{-im}(x))}\| \leq \lambda^n)$$

holds for any $n \geq 1$. If x is both an (m,λ)-E-Pliss point and an (m,λ)-F-Pliss point, then it is called an *(m,λ)-bi-Pliss point.* Two (m,λ)-E-Pliss points $(f^k(x), f^l(x))$ on an orbit $\mathrm{Orb}(x)$ are called *consecutive (m,λ)-E-Pliss points*, if $k < l$ and $f^i(x)$ is not an (m,λ)-E-Pliss point for any $k < i < l$. Similarly we define *consecutive (m,λ)-F-Pliss points.* In particular, when $m = 1$, we just call λ-E-Pliss point, λ-F-Pliss point or λ-*bi*-Pliss point for short.

Remark 2.17. *It is well known that the stable manifold of a λ-E-Pliss point has a uniform scale of dimension $\dim(E)$ which depends only on the diffeomorphism f, see for example* [1].

Definition 2.18. Consider a diffeomorphism $f \in \mathrm{Diff}^1(M)$ and a constant $0 < \lambda < 1$. An invariant compact set K with a dominated splitting $T_K M = E \oplus F$ is called a *λ-E-weak set* (resp. λ-F-weak set), if there is no λ-E-Pliss point (resp. λ-F-Pliss point) contained in K.

We have the following lemma stating that for an invariant compact set admitting a dominated splitting, if one of the two bundles is weak enough, then the other bundle is uniformly hyperbolic.

Lemma 2.19. *Given a diffeomorphism $f \in \mathrm{Diff}^1(M)$, consider an invariant compact set K which admits a $(1,\lambda^2)$-dominated splitting $T_K M = E \oplus F$. For any constant $\lambda' \in (\lambda, 1)$, assume that K is a λ'-F-weak set, then $E|_K$ is uniformly contracted. Similarly, if K is a λ'-E-weak set, then $F|_K$ is uniformly expanded.*

Proof. By Definitions 2.16 and 2.18, for any point $y \in K$, there is an integer $n_y \geq 1$, such that

$$\prod_{i=0}^{n_y-1} \|Df^{-1}|_{F(f^{-i}(y))}\| > (\lambda')^{n_y}.$$

Then one has that

$$\prod_{i=0}^{n_y-1} \|Df|_{E(f^i(f^{-n_y}(y)))}\| < \left(\frac{\lambda^2}{\lambda'}\right)^{n_y} < \lambda^{n_y}.$$

By the compactness of the set K, the integers n_y are uniformly bounded. Hence by a standard argument, one can see that $E|_K$ is uniformly contracted. □

One can obtain Pliss points by the following lemma given by V. Pliss, see [61, 64].

Lemma 2.20 (Pliss lemma)**.** *Assume that K is an invariant compact set of a diffeomorphism f in $\mathrm{Diff}^1(M)$ and E is an invariant sub-bundle of $T_K M$. For any two numbers $0 < \lambda_1 < \lambda_2 < 1$, we have:*

1. *There are a positive integer $N = N(\lambda_1, \lambda_2, f)$ and a number $c = c(\lambda_1, \lambda_2, f)$ such that for any $x \in K$ and any number $n \geq N$, if*

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i x)}\| \leq {\lambda_1}^n,$$

then there are $0 \leq n_1 < n_2 < \cdots < n_l \leq n$ such that $l \geq cn$, and, for any $j = 1, \cdots, l$ and any $k = n_j + 1, \cdots, n$,

$$\prod_{i=n_j}^{k-1} \|Df|_{E(f^i x)}\| \leq {\lambda_2}^{k-n_j}.$$

2. *For any point $x \in K$, and any integer m, if for all $n \geq m$,*

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i x)}\| \leq {\lambda_1}^n,$$

then there is an infinite sequence $0 \leq n_1 < n_2 < \cdots$, such that

$$\prod_{i=n_j}^{k-1} \|Df|_{E(f^i x)}\| \leq {\lambda_2}^{k-n_j},$$

for all $k > n_j$ and all $j = 1, 2, \cdots$.

Corollary 2.21. *Consider a diffeomorphism $f \in \mathrm{Diff}^1(M)$ and an invariant compact set K which admits a dominated splitting $T_K M = E \oplus F$. For any $x \in K$, we have that:*

1. *If x is an (m, λ)-E-Pliss point, then there is a point $y \in \omega(x)$, such that y is also a (m, λ)-E-Pliss point.*
2. *If for any $y \in \omega(x)$, there is an integer $n_y \in \mathbb{N}$, such that*

$$\prod_{i=0}^{n_y-1} \|Df^m|_{E(f^{im}(y))}\| \leq \lambda^{n_y},$$

then for any $\lambda' \in (\lambda, 1)$, there are infinitely many (m, λ')-E-Pliss points on $\mathrm{Orb}^+(x)$.

Proof. By considering the diffeomorphism f^m instead of f, we can assume that $m = 1$. The proof of the general case is similar.

(1) By item 2 of Pliss lemma, for any $\lambda' \in (\lambda, 1)$, there are infinitely many λ'-E-Pliss points on $\mathrm{Orb}^+(x)$. Take a limit point of these λ'-E-Pliss points, denote it by $y_{\lambda'}$, then $y_{\lambda'} \in \omega(x)$ is a λ'-E-Pliss point. We take a sequence of numbers $(\lambda_n)_{n\geq 1}$ such that $\lambda_n \in (\lambda, 1)$ and $\lambda_n \to \lambda$ when n goes to infinity. Then for any $n \geq 1$, there is a λ_n-E-Pliss point $y_{\lambda_n} \in \omega(x)$. Taking a subsequence if necessary, we assume $(y_{\lambda_n})_{n\geq 1}$ converges to a point $y \in \omega(x)$. Then y is a λ_n-E-Pliss point for any $n \geq 1$. Since $\lambda_n \to \lambda$, the point y is a λ-E-Pliss point.

(2) By the compactness of $\omega(x)$, there is an integer N, such that $n_y \leq N$ for any $y \in \omega(x)$. There is a constant $C > 0$, such that, for any $y \in \omega(x)$, we have for any $n \geq 1$,

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i(y))}\| < C\lambda^n.$$

Take a constant $\lambda' \in (\lambda, 1)$. Take three constants $\lambda_1 < \lambda_2 < \lambda_3$ contained in (λ, λ'). There is $N \in \mathbb{N}$, such that $C\lambda^n < \lambda_1^n$ for any $n \geq N$. There is $\varepsilon > 0$, such that, for any two points $x_1, x_2 \in K$, if $d(f(x_1), f(x_2)) < \varepsilon$, then $\frac{\|Df|_{E(f^i(x_1))}\|}{\|Df|_{E(f^i(x_2))}\|} < \frac{\lambda_2}{\lambda_1}$, for all $i = 0, 1, \cdots, N$. By considering an iterate of x instead of x, we can assume that $d_H(\overline{\mathrm{Orb}^+(x)}, \omega(x)) < \varepsilon$, where $d_H(\cdot, \cdot)$ is the Hausdorff distance. Then for any $n \geq 1$, we have

$$\prod_{i=0}^{nN} \|Df|_{E(f^i(x))}\| < (C\lambda^N)^n \left(\frac{\lambda_2}{\lambda_1}\right)^{nN} < \lambda_2^{nN}.$$

There is $T > 0$, such that, for any $k \geq T$, we have $\lambda_2^{kN} \|Df\|^j < \lambda_3^{kN+j}$ for all $j = 0, 1, \cdots, N-1$. Then for any $n > TN$, assume $n = kN + j$, where $0 \leq j < N$, we have

$$\prod_{i=0}^{n} \|Df|_{E(f^i(x))}\| \leq \left(\prod_{i=0}^{kN} \|Df|_{E(f^i(x))}\|\right) \|Df\|^j < \lambda_2^{kN} \|Df\|^j < \lambda_3^n.$$

Then by item 2 of Pliss lemma, there are infinitely many (m, λ')-E-Pliss points on $\mathrm{Orb}^+(x)$. □

For Pliss-points, we have the following lemma. The technics of the proof can be found in many papers, for example [64, 65].

Lemma 2.22. *Assume K is an invariant compact set of a diffeomorphism $f \in \mathrm{Diff}^1(M)$ with an (m, λ^2)-dominated splitting $T_K M = E \oplus F$. We have that, for any $\lambda' \in (\lambda, 1)$:*

1. *If a sequence of consecutive (m, λ')-E-Pliss points $(f^{n_i}(x_i), f^{l_i}(x_i))_{i \geq 0}$ satisfies that $l_i - n_i \to +\infty$, then, take any limit point y of the sequence $(f^{l_i}(x_i))$, we have that y is a (m, λ')-bi-Pliss point.*

2. *If there are both (m, λ')-E-Pliss points for f on $\mathrm{Orb}^+(x)$ and (m, λ')-F-Pliss points on $\mathrm{Orb}^-(x)$, then there is at least one (m, λ')-bi-Pliss point on $\mathrm{Orb}(x)$.*

3. *If $x \in K$ is an (m, λ')-E-Pliss point and there are no other (m, λ')-E-Pliss points on $\mathrm{Orb}^-(x)$, then x is also an (m, λ)-F-Pliss point. Thus x is an (m, λ')-bi-Pliss point.*

We have the following selecting lemma of Liao to get weak periodic orbits (see [52], [75]).

Lemma 2.23 (Liao's selecting lemma)**.** *Assume $f \in \mathrm{Diff}^1(M)$. Consider an invariant compact set Λ with a non-trivial (m, λ)-dominated splitting $T_\Lambda M = E \oplus F$, and $\lambda_0 \in (\lambda, 1)$, if the following two conditions are satisfied:*

- *The bundle E is not uniformly contracted.*

- *For any invariant compact subset* $K \subsetneqq \Lambda$, *there is an* (m, λ_0)*-E-Pliss point* $x \in K$.

Then for any neighborhood U *of* Λ, *for any* $\lambda_1 < \lambda_2$ *contained in* $(\lambda_0, 1)$, *there is a periodic orbit* $\mathrm{Orb}(q) \subset U$ *with period* $\tau(q)$ *a multiple of* m, *such that, for all* $n = 1, \cdots, \tau(q)/m$, *the following two inequalities are satisfied:*

$$\prod_{i=0}^{n-1} \|Df^m|_{E(f^{im}(q))}\| \leq {\lambda_2}^n,$$

and

$$\prod_{i=n-1}^{\tau(q)/m-1} \|Df^m|_{E(f^{im}(q))}\| \geq {\lambda_1}^{\tau(q)/m-n+1}.$$

Particularly, one can find a sequence of periodic points that are homoclinic related with each other and converges to a point in Λ. *Similar assertions for* F *hold with respect to* f^{-1}.

2.5 Perturbation technics

Consider a diffeomorphism f and a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$. We call two perturbations f_1 and f_2 have *disjoint support*, if $f_i|_{M\setminus U_i} = f|_{M\setminus U_i}$ for $i = 1, 2$, and $U_1 \cap U_2 = \emptyset$. In general, the diffeomorphism g, where $g|_{M\setminus(U_1\cup U_2)} = f|_{M\setminus(U_1\cup U_2)}$ and $g|_{U_i} = f_i|_{U_i}$ for $i = 1, 2$, is not contained in $\mathcal{U}$ any more. However, there is a basis of neighborhoods $\mathcal{U}$ of f, such that if the element of $\mathcal{U}$ is of the form $f \circ \phi$ with $\phi \in \mathcal{V}$, where $\mathcal{V}$ is a C^1 neighborhood of Id, then $\mathcal{V}$ satisfies the following property (F), see Section 2 of [63].

Definition 2.24 (Property (F))**.** Assume $\mathcal{V}$ is a C^1 neighborhood of Id. We say $\mathcal{V}$ satisfies the *property (F)*, if for any perturbations ϕ and ϕ' of Id in $\mathcal{V}$ with disjoint support, the composed perturbation $\phi \circ \phi'$ is still in $\mathcal{V}$.

We give some tools for C^1-perturbation. First is the famous Hayashi's connecting lemma, see [49, 77].

Theorem 2.25. *Assume* f *is a diffeomorphism in* $\mathrm{Diff}^1(M)$. *For any neighborhood* $\mathcal{U}$ *of* f, *there is an integer* $N \in \mathbb{N}$, *satisfying the following property: For any point* x *that is not a periodic point of* f *with period less than or equal to* N, *for any neighborhood* V_x *of* x *there is a neighborhood* $V'_x \subset V_x$, *such that, for any two points* $p, q \notin \bigcup_{i=0}^{N-1} f^i(V_x)$, *if* p *has a positive iterate* $f^{n_p}(p) \in V'_x$ *and* q *has a negative iterate* $f^{-n_q}(q) \in V'_x$, *where* $n_p, n_q \in \mathbb{N}$, *then there is a diffeomorphism* $g \in \mathcal{U}$ *that coincides with* f *outside* $\bigcup_{i=0}^{N-1} f^i(V_x)$ *and* q *is on the positive orbit of* p. *Moreover, assume* $g^m(p) = q$, *then* $m \leq n_p + N + n_q$ *and* $\{p, g(p), \cdots, g^m(p) = q\} \subset \bigcup_{i=0}^{n_p}\{f^i(p)\} \cup \bigcup_{i=0}^{N-1} f^i(V_x) \cup \bigcup_{i=0}^{n_q}\{f^{-i}(q)\}$.

Theorem 2.25 deals with a single diffeomorphism and a given neighborhood. Here we give a uniform version that is valid to a neighborhood of a diffeomorphism, see [74].

Theorem 2.26 (A uniform connecting lemma, Theorem A of [74])**.** *Assume that f is a diffeomorphism in $\mathrm{Diff}^1(M)$. For any C^1 neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, there are three numbers $\rho > 1$, $\delta_0 > 0$ and $N \in \mathbb{N}$, together with a C^1 neighborhood $\mathcal{U}_1 \subset \mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, that satisfy the following property: For any $f_1 \in \mathcal{U}_1$, any point $z \in M$ and any number $0 < \delta < \delta_0$, as long as the N balls $(f_1^i(B(z,\delta)))_{0\leq i\leq N-1}$ are pairwise disjoint and each is of size smaller than δ_0 (that is to say, $f_1^i(B(z,\delta)) \subset B(f_1^i(z),\delta_0)$), then for any two points x and y that are outside the set $\Delta = \bigcup_{0\leq i\leq N-1} f_1^i(B(z,\delta))$, if there are two positive integers n_x and n_y such that $f_1^{n_x}(x) \in B(z,\delta/\rho)$ and $f_1^{-n_y}(y) \in B(z,\delta/\rho)$, then there are a diffeomorphism $g \in \mathcal{U}$ and a positive integer m such that $g^m(x) = y$ and $g = f_1$ off Δ. Moreover, the piece of orbit $\{x, g(x), \cdots, g^m(x) = y\}$ is contained in the set $\{x, f_1(x), \cdots, f_1^{n_x}(x)\} \cup \Delta \cup \{y, f_1^{-1}(y), \cdots, f_1^{-n_y}(y)\}$ and the number m is no more than $n_x + n_y$.*

To control the perturbing neighborhood when connecting two points that are close, we have the following lemma, see [8], also see [63, Theorem 6.1], and [77, Lemma 2.1].

Lemma 2.27 (Basic perturbation lemma)**.** *For any neighborhood $\mathcal{U}$ of a diffeomorphism $f \in \mathrm{Diff}^1(M)$, there are two numbers $\theta > 1$ and $r_0 > 0$ satisfying: for any two points $x, y \in M$ contained in a ball $B(z,r)$, where $r \leq r_0$, there is a diffeomorphism $g \in \mathcal{U}$, such that $g(x) = f(y)$, and g coincides with f outside the ball $B(z, \theta \cdot r)$.*

Definition 2.28. For a chart $\varphi : V \to \mathbb{R}^d$ of M, a set C is called a *cube* of φ if $\varphi(C)$ is the image of $[-a,a]^d$ by a translation of $\mathbb{R}^d$, where a is the radius of the cube. If a cube with radius $(1+\varepsilon)a$ and the same center of $\varphi(C)$ is still contained in $\varphi(V)$, we denote by $(1+\varepsilon)C$ its pre-image of φ.

Definition 2.29. Consider a chart $\varphi : V \to \mathbb{R}^d$. A *tiled domain* according to the chart of φ is an open set $U \subset V$ and a family $\mathcal{C}$ of cubes of φ (called *tiles* of domain), such that:

1. the interior of the tiles are pairwise disjoint;
2. the union of all tiles of $\mathcal{C}$ equals to U;
3. the geometry of the tiling is bounded, i.e.

- the number of tiles around each point is uniformly bounded (by 2^d), that is to say, there is a neighborhood for each point that meets at most 2^d tiles,

– for any two pairs (C, C') of intersecting tiles, the rate of their diameters is uniformly bounded (by 2).

By a standard construction, any open set $U \subset V$ can be tiled according to the coordinates of φ (e.g. [15, 35]).

Definition 2.30. Assume $f \in \mathrm{Diff}^1(M)$. Consider a neighborhood $\mathcal{U} \subset \mathrm{Diff}^1(M)$ and a number N. A tiled domain $(U, \mathcal{C})$ is called a *perturbation domain* of order N of $(f, \mathcal{U})$, if the following properties are satisfied.

1. U is disjoint from its N first iterates of f.

2. For any finitely many sequence of pairs of points $\{(x_i, y_i)\}_{1 \leq i \leq l}$ in U, such that for any $i = 1, 2, \cdots, l$, the points x_i and y_i are contained in the same tile of $\mathcal{C}$, then there exist:

   - a diffeomorphism $g \in \mathcal{U}$, that coincides with f outside $\bigcup_{0 \leq i \leq N-1} f^i(U)$,
   - a strictly increasing sequence $1 = n_0 < n_1 < \cdots < n_k \leq l$, such that $g^N(x_{n_i}) = f^N(y_{n_{i+1}-1})$ for any $i \neq k$, and $g^N(x_{n_k}) = f^N(y_l)$.

The union $\bigcup_{0 \leq i \leq N-1} f^i(U)$ is called the *support* of the perturbation domain $(U, \mathcal{C})$ and denoted by $\mathrm{Supp}(U)$.

Definition 2.31. A pseudo-orbit $(x_0, x_1, \cdots, x_l)$ is said to *keep the tiles* of a perturbation domain $(U, \mathcal{C})$ of order N of $(f, \mathcal{U})$, if the intersection of the pseudo-orbit and $\mathrm{Supp}(U)$ is a union of segments $x_{n_i}, x_{n_i+1}, \cdots, x_{n_i+N-1}$ of the form that $x_{n_i} \in U$ and for any $j = 1, 2, \cdots, N-1$, $x_{n_i+j} = f^j(y_{n_i})$, where y_{n_i} is a point contained in the same tile of $\mathcal{C}$ as x_{n_i}. A pseudo-orbit $(x_0, x_1, \cdots, x_k)$ is said to *have jumps only in tiles* of a perturbation domain $(U, \mathcal{C})$ of order N of $(f, \mathcal{U})$, if it keeps the tiles and for any $x_i \notin \mathrm{Supp}(U)$, we have $x_{i+1} = f(x_i)$. For a family of perturbation domains $(U_k, \mathcal{C}_k)_{k \geq 0}$ of order N_k of $(f, \mathcal{U}_k)$ with disjoint support, we say that a pseudo-orbit $(x_0, x_1, \cdots, x_l)$ has *jumps only in tiles* of the perturbation domains $(U_k, \mathcal{C}_k)_{k \geq 0}$, if it keeps the tiles of the perturbation domains and for any $x_i \notin \bigcup_k \mathrm{Supp}(U_k)$, we have $x_{i+1} = f(x_i)$.

By the proof of connecting lemma in [9], the perturbation domain always exists (see also [15, Théorème 2.1] and [35, Théorème 3.3]).

Theorem 2.32 (Another statement of the connecting lemma)**.** *For any neighborhood $\mathcal{U}$ of f, there is an integer $N \geq 1$, and for all point $p \in M$, there is a chart $\varphi : V \to \mathbb{R}^d$ such that any tiled domain $(U, \mathcal{C})$ according to φ disjoint from its N first iterates is a perturbation domain of order N for $(f, \mathcal{U})$.*

From the definitions above, we can get the following lemma easily.

Lemma 2.33 (Lemme 2.3 of [15]). *For a family of disjoint perturbation domains $(U_k, \mathcal{C}_k)$ of order N_k of $(f, \mathcal{U}_k)$ with disjoint support, if there is a pseudo-orbit $(p = p_0, p_1, \cdots, p_m = q)$ that has only jumps in the tiles of $(U_k, \mathcal{C}_k)_{k\geq 0}$ and $p_0, p_m \notin U_k \cup \cdots \cup f^{N_k-1}(U_k)$ for all $k \geq 0$, then for any i, there is $g_i \in \mathcal{U}_i$ and a new pseudo-orbit $(p = p'_0, \cdots, p'_{m'} = q)$ of g_i that has only jumps in the tiles of domains $(U_k, \mathcal{C}_k)_{k\geq 0, k\neq i}$. Moreover, $g_i = f$ outside $U_i \cup \cdots \cup f^{N_i-1}(U_i)$ and $\{p'_0, \cdots, p'_{m'}\} \setminus (U_i \cup \cdots \cup f^{N_i-1}(U_i)) \subset \{p_0, p_1, \cdots, p_m\}$, and $m' \leq m$.*

2.6 Topological towers

In this subsection, we introduce two lemmas of [15] that are useful to get a true orbit by perturbing a pseudo-orbit. These two lemmas are the key tools in the proof of Proposition 3.6. First we give the following lemma to choose perturbation neighborhoods. In fact, it is a generalized result of Lemme 3.7 of [15], but one can get the conclusion from the proof in [15].

Lemma 2.34. *There is a constant $\kappa_d > 0$ (which only depends on the dimension d of M) satisfying the following property.*
Consider an integer $T > 0$. Assume that W' and V' are two compact d-dimensional sub-manifolds with boundary, satisfying V' is disjoint from its $\kappa_d T$ first iterates. Then for any neighborhood U_1 of W' and any neighborhood U_2 of V', there are two open sets W and V, satisfying the following properties.

1. *$\overline{W}$ and $\overline{V}$ are two compact d-dimensional sub-manifolds with boundary;*
2. *$W' \subset W \subset \overline{W} \subset U_1$ and $V' \subset \bigcup_{i=0}^{\kappa_d T} f^{-i}(V)$;*
3. *$\overline{V}$ is contained in $U_2 \cup f(U_2) \cup \cdots \cup f^{\kappa_d T}(U_2)$ and disjoint from its N first iterates.*
4. *$\overline{W} \cap (\bigcup_{i=-T}^{T} f^i(\overline{V})) = \emptyset$.*

Remark 2.35. (1) *We point out here that, Lemme 3.7 of [15] assumes more that W' is disjoint from its T first iterates, and in the conclusion the set $W \cup V$ is disjoint from its T first iterates. Moreover, the statement of Lemme 3.7 of [15] does not involve the two neighborhoods U_1 and U_2. But the proof of Lemme 3.7 of [15] gives all the information stated in Lemma 2.34, see [15, Page 61–62].*

(2) *In Lemma 2.34, if we assume more that W' is disjoint with its first T' iterates where $T' \leq T$, then by taking U_1 small enough, we can obtain that W is disjoint from its T_1 first iterates, and the union $W \cup V$ is also disjoint from its T_1 first iterates.*

Then, we give a lemma of [15] for the construction of what they called *topological tower* (see Théorème 3.1 and Corollaire 3.1 in [15]). Denote by $Per_T(f)$ the set of periodic orbits with period less than T.

Lemma 2.36 (Topological tower)**.** *There is a constant $\kappa_d > 0$ (which only depends on the dimension d of M), such that, for any $T \in \mathbb{N}$, any constant $\delta > 0$, any compact set K of $f \in \mathrm{Diff}^1(M)$ that does not contain any non-hyperbolic periodic orbits with periods less than $\kappa_d N$ and any neighborhood U of K, there exist an open set V and a compact set $D \subset V$, satisfying the following properties:*

1. *The closure of V is a compact d-dimensional sub-manifold with boundary.*
2. *For any point $x \in K$ with $x \notin \bigcup_{p \in Per_T(f)} W^s_\delta(p)$, there is $n > 0$, such that $f^n(x) \in \mathrm{Int}(D)$.*
3. *The sets $\overline{V}, f(\overline{V}), \cdots, f^T(\overline{V})$ are pairwise disjoint.*
4. *The set $\overline{V}$ is contained in $U \cup f(U) \cup \cdots \cup f^{\kappa_d T}(U)$.*

Moreover, the diameter of all connected components of V can be arbitrarily small.

Remark 2.37. (1) *In [15], Théorème 3.1 is stated for an invariant compact set K, and the items 1 and 4 in the conclusion of Lemma 2.36 are not stated. But from the proof (see [15, Page 62–63]), we can see that the conclusion is also true for non-invariant compact sets and also the items 1 and 4 are true.*

(2) *We give a sketch of the proof of Lemma 2.36. Take κ_d to be the constant in Lemma 2.34. First, one can take a compact sub-manifold U_0 of M with boundary that is disjoint from its first N iterates, such that, any point in a small neighborhood O of $Per_T(f)$ that is not on the local stable manifold of $Per_T(f)$ has a positive iterate in the interior of U_0. Then one can take a finite cover of the compact set $K \setminus O$ by open sets $\{V_i\}_{0 \le i \le r}$ that are disjoint from their first $\kappa_d T$ iterates (they are not disjoint from each other in general). Moreover, for each $0 \le i \le r$ the closure of V_i is a compact d-dimensional sub-manifold with boundary. Then one can apply Lemma 2.34 inductively to $T_1 = T_2 = T$, U_i and V_i, considering U_i and V_i to be W' and V' respectively, and obtain U_{i+1} as S in Lemma 2.34. Moreover, since in this setting, U_0 is disjoint from its first T iterates, one can obtain that U_i is disjoint from its first T iterates for each $1 \le i \le r+1$. Finally, one can take V to be the interior of U_{r+1}. For more details, the reader should refer to [15, Page 62–63].*

2.7 Sufficient conditions for existence of an ergodic measure

We define a relationship called *multiple almost shadowing* between two periodic orbits, which was called good approximation in [22, 41].

Definition 2.38 (Multiple almost shadowing)**.** Consider two periodic orbits P and Q of a map $f : M \to M$ and two numbers $\gamma > 0$ and $0 < \varkappa \leq 1$. Denote by $\pi(P)$ the period of P. We say P is $(\gamma, \varkappa)$-*multiple almost shadowed* by Q, if there are a subset $\Gamma \subset Q$ and a map $\rho : \Gamma \to P$, that satisfy the following properties.

- $\frac{\#\Gamma}{\#Q} \geq \varkappa$.
- $\#\rho^{-1}(f^j(p))$ is constant, for any $p \in P$.
- $d(f^j(q), f^j(\rho(q))) < \gamma$, for any $q \in \Gamma$ and for any $j = 0, 1, \cdots, \pi(P) - 1$.

One says a periodic point p has *simple spectrum*, if the d Lyapunov exponents of $\mathrm{Orb}(p)$ are mutually different. The following lemma is standard, see a similar statement in [22, Theorem 3.5].

Lemma 2.39. *For generic $f \in \mathrm{Diff}^1(M)$, consider a periodic point p with simple spectrum whose homoclinic class $H(p)$ is non-trivial. Then for any $\varepsilon, \gamma > 0$, and any $\varkappa \in (0, 1)$, there is a periodic point q with simple spectrum homoclinically related to p, such that the following properties are satisfied:*

1. $\mathrm{Orb}(q)$ *is ε-dense in $H(p)$,*
2. $\mathrm{Orb}(p)$ *is $(\gamma, \varkappa)$-multiple almost shadowed by* $\mathrm{Orb}(q)$.
3. $\chi_i(q, f)$ *is ε-close to $\chi_i(p, f)$, for any $i = 1, 2, \cdots, d$.*

In particular, if p is center-dissipative, then q can be chosen to be center-dissipative.

Sketch of the proof. The properties in the statement are persistent under C^1 perturbations. Hence we only show that one can obtain such a periodic point q by C^1-small perturbations, and by a standard Baire argument, the statement holds for generic systems.

Since p has simple spectrum, there is a dominated splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_d$ over $\mathrm{Orb}(p)$, such that, each E_i is the one dimensional sub-bundle corresponding to the i^{th} Lyapunov exponent of $\mathrm{Orb}(p)$. Take a transverse homoclinic point $x \in W^s(p) \cap W^u(p)$, such that $\mathrm{Orb}(x) \cup \mathrm{Orb}(p)$ is $\frac{\varepsilon}{2}$-dense in $H(p)$. By similar technics to the proof of [34, Proposition 1.10, Page 689], arbitrarily C^1-close to f in $\mathrm{Diff}^1(M)$, there is a diffeomorphism g, which coincides with f on $\mathrm{Orb}(x)$ and outside a small neighborhood of the point x, such that the dominated splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_d$ can be spread to the set $\mathrm{Orb}(x) \cup \mathrm{Orb}(p)$. Since $\mathrm{Orb}(x) \cup \mathrm{Orb}(p)$ is still a hyperbolic set with respect to g, by the shadowing lemma, there is a hyperbolic periodic point q homoclinically related to p, such that $\mathrm{Orb}(q)$ spends an arbitrarily large portion of time close to $\mathrm{Orb}(p)$, and $\frac{\varepsilon}{2}$-shadows $\mathrm{Orb}(x) \cup \mathrm{Orb}(p)$. Then the items 1, 2 are satisfied. Since the dominated splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_d$ with each bundle of dimension one spreads to the set $\mathrm{Orb}(x) \cup \mathrm{Orb}(q)$, each Lyapunov exponent $\chi_i(q, g)$ can be presented as $\frac{1}{\pi(q,g)} \log \|Dg^{\pi(q,g)}|_{E_i(q)}\|$. Then the item 3 can be obtained by the fact that $\mathrm{Orb}(q)$ spends most of the time close to $\mathrm{Orb}(p)$. $\square$

The following lemma is proved in [22] to obtain ergodic measures, using the method developed in [46], by taking the weak-$*$-limit of atomic measures supported on periodic orbits.

Lemma 2.40 (Lemma 2.5 of [22]). *Consider two sequences of numbers $(\gamma_n)_{n\geq 1}$ in $(0, +\infty)$, and $(\varkappa_n)_{n\geq 1}$ in $(0, 1]$, such that $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\prod_{n=1}^{\infty} \varkappa_n > 0$. Assume $(p_n)_{n\geq 1}$ is a sequence of periodic points of a map $f : M \to M$ with increasing periods $\pi(p_n)$, and denote by μ_n the probability atomic measure uniformly distributed on the orbit of p_n. If* $\mathrm{Orb}(p_n)$ *is $(\gamma_n, \varkappa_n)$-multiple almost shadowed by* $\mathrm{Orb}(p_{n+1})$ *for any $n \geq 1$, then the sequence of measures (μ_n) converges to an ergodic measure μ and* $\mathrm{supp}(\mu) = \bigcap_{n=1}^{\infty}(\overline{\bigcup_{k=n}^{\infty} \mathrm{Orb}(p_k)})$.

2.8 Perturbation lemmas about periodic cocycles

Consider a family of linear maps $A_1, \cdots, A_n \in GL(d, \mathbb{R})$. Denote by $B = A_n \circ \cdots \circ A_1$ and denote by $\lambda_1(B), \cdots, \lambda_d(B)$ the eigenvalues of B counted by multiplicity such that $|\lambda_1| \leq \cdots \leq |\lambda_d|$. Then the i^{th} *Lyapunov exponent* of B is defined as $\chi_i(B) = \frac{1}{n} \log |\lambda_i(B)|$.

The following statement follows from [11], which allows to modify only two consecutive Lyapunov exponents of a cocycle, see also [17, Lemma 4.4]. A similar result can also be found in [43, Theorem 1.2].

Lemma 2.41 ([11],Theorem 4.1 and Proposition 3.1). *For any constants $D > 1$, $\varepsilon > 0$ and $d \geq 2$, there are two constants T and n_0 satisfying the following property.*

Consider a family of linear maps $A_1, \cdots, A_n \in GL(d, \mathbb{R})$ with $n \geq n_0$, such that $\|A_i\|, \|A_i^{-1}\| \leq D$, and the linear map $B = A_n \circ \cdots \circ A_1$ has no T-dominated splitting of index i for some $i \in \{1, \cdots, d-1\}$. Then there exist one-parameter families of linear maps $(A_{m,t})_{t\in[0,1]}$ for any $1 \leq m \leq n$, satisfying the following properties.

1. *$A_{m,0} = A_m$, for any $1 \leq m \leq n$.*

2. *$\|A_{m,t} - A_m\| < \varepsilon$ and $\|A_{m,t}^{-1} - A_m^{-1}\| < \varepsilon$, for any $t \in [0, 1]$ and for any $1 \leq m \leq n$.*

3. *Consider the linear map $B_t = A_{n,t} \circ \cdots \circ A_{1,t}$, then the Lyapunov exponents of B_t satisfy the following properties.*

- $\chi_j(B_t) = \chi_j(B)$ *for any* $j \neq i, i+1$,
- $\chi_i(B_t) + \chi_{i+1}(B_t) = \chi_i(B) + \chi_{i+1}(B)$,
- $\chi_i(B_t)$ *is non-decreasing and* $\chi_{i+1}(B_t)$ *is non-increasing, that is*

$$\chi_i(B_t) \leq \chi_i(B_{t'}) \leq \chi_{i+1}(B_{t'}) \leq \chi_{i+1}(B_t), \text{ for any } t < t',$$

- $\chi_i(B_1) = \chi_{i+1}(B_1)$.

We state a generalized version of Franks' Lemma [42], which is proved in [48].

Lemma 2.42 ([48]). *Consider a diffeomorphism* $f \in \mathrm{Diff}^1(M)$, *a hyperbolic periodic point* q *of period* π *and a constant* $\varepsilon > 0$. *Assume that for any* $n = 0, 1, \cdots, \pi - 1$, *there is a one-parameter family of linear maps* $(A_{n,t})_{t\in[0,1]}$ *in* $GL(d, \mathbb{R})$, *such that the following properties are satisfied:*

- $A_{n,0} = Df(f^n(q))$,
- $\|Df(f^n(q)) - A_{n,t}\| < \varepsilon$ *and* $\|Df^{-1}(f^{n+1}(q)) - A_{n,t}^{-1}\| < \varepsilon$, *for any* $t \in [0, 1]$,
- $A_{\pi-1,t} \circ \cdots \circ A_{0,t}$ *is hyperbolic for any* $t \in [0, 1]$,.

Then, for any neighborhood V *of* $\mathrm{Orb}(q)$, *any constant* $\delta > 0$, *and any pair of compact sets* $K^s \subset W^s_\delta(q, f)$ *and* $K^u \subset W^u_\delta(q, f)$ *that are disjoint from* V, *there is a diffeomorphism* g *that is* ε*-close to* f *in* $\mathrm{Diff}^1(M)$, *and that satisfies the following properties:*

1. g *coincides with* f *on* $M \setminus V$ *and* $\mathrm{Orb}(q)$;
2. $K^s \subset W^s_\delta(q, g)$ *and* $K^u \subset W^u_\delta(q, g)$;
3. $Dg(g^n(q)) = Dg(f^n(q)) = A_{n,1}$ *for all* $n = 0, \cdots, \pi - 1$.

2.9 Some previous results

In this section, we give some previous results, which are applied in the proofs of the theorems.

The following result is a combination of Proposition 7.1 and Proposition 8.1 of [17].

Proposition 2.43. *Consider a diffeomorphism* $f \in \mathrm{Diff}^1(M)$, *and a hyperbolic periodic point* p *of index* i, *where* $2 \leq i \leq d-1$. *Assume the following properties are satisfied:*

- $H(p)$ *has no dominated splitting of index* $i - 1$,

– *for any* $\varepsilon > 0$*, there is a periodic point* p_ε *homoclinically related to the orbit of* p*, such that* $\chi_i(p_\varepsilon) \in (-\varepsilon, 0)$*.*

Then for any C^1*-neighborhood* $\mathcal{U}$ *of* f*, there is a diffeomorphism* $g \in \mathcal{U}$ *having a heterodimensional cycle associated with* $\mathrm{Orb}(p_g)$ *and a periodic point* q_g *of* g *of index* $i-1$*.*

The next result is a combination of [22, 41].

Theorem 2.44 ([22, 41])**.** *For generic* $f \in \mathrm{Diff}^1(M)$*, assume that* p *is a hyperbolic periodic point of index* i*. If the homoclinic class* $H(p)$ *contains a hyperbolic point* q *of index smaller than* i*, then there is an ergodic measure supported on* $H(p)$ *whose* i^{th} *Lyapunov exponent is* 0*. Moreover, if* $\mathrm{Ind}(q) = i-1$ *and there is a dominated splitting* $T_{H(p)}M = E \oplus F \oplus G$ *such that* $\dim(E) = i-1$ *and* $\dim(F) = 1$*, then, there is an ergodic measure* μ *such that* $\mathrm{supp}(\mu) = H(p)$ *and the* i^{th} *Lyapunov exponent of* μ *is* 0*.*

Recall that for a hyperbolic ergodic measure μ, there is a full μ-measure set Λ, such that, there is a splitting $T_\Lambda M = \mathcal{E}^s \oplus \mathcal{E}^u$ such that, the bundles $\mathcal{E}^s$ and $\mathcal{E}^u$ are associated to the negative and positive Lyapunov exponents respectively. We call this splitting the *non-uniform hyperbolic splitting* of μ and the dimension of $\mathcal{E}^s$ is called the *index* of μ. The following result claims the support of a hyperbolic measure intersects a homoclinic class if the above splitting is a dominated splitting.

Proposition 2.45 (Proposition 1.4 of [34])**.** *Let* μ *be a hyperbolic ergodic measure of index* i*. If the non-uniform hyperbolic splitting of* μ *is a dominated splitting, then there is a hyperbolic periodic point* p *of index* i*, such that* $\mathrm{supp}(\mu) \cap H(p) \neq \emptyset$*. Moreover, if* μ *is ergodic, then* $\mathrm{supp}(\mu) \subset H(p)$*.*

2.10 Generic properties

A set R of a topological Baire space X is called a *residual* set, if R contains a dense G_δ set of X. We say a property is a *generic* property of X, if there is a residual set $R \subset X$, such that each element contained in R satisfies the property. We give some well known C^1-generic properties of diffeomorphisms in the following theorem. These results can be found in many papers like [3, 15, 21, 23, 32, 41, 62].

Theorem 2.46. *There is a residual set* $\mathcal{R}$ *in* $\mathrm{Diff}^1(M)$ *of diffeomorphisms, such that any* $f \in \mathcal{R}$ *satisfies the following properties:*

1. *The diffeomorphism* f *is Kupka-Smale: all periodic points are hyperbolic and the stable and unstable manifolds of any two periodic orbits intersect transversely. Moreover, every periodic point is center-dissipative with respect to* f *or to* f^{-1}*.*

2. *The periodic points are dense in the chain recurrent set and any any chain recurrence class containing a periodic point coincides with its homoclinic class, hence two homoclinic classes either coincide or are disjoint.*

3. *For any two points $x, y \in M$ and any open set W, if $x \dashv y$ then $x \prec y$, and if $x \dashv_W y$, then $x \prec_W y$.*

4. *For a periodic point p of f, there exists a C^1-neighborhood $\mathcal{U}_1$ of f, such that every $g \in \mathcal{U}_1 \cap \mathcal{R}$ is a continuity point for the map $g \mapsto H(p_g, g)$ where p_g is the continuation of p for g, where the continuity is with respect to the Hausdorff distance between compact subsets of M.*

5. *If $H(p)$ is a homoclinic class of f, then there exists an interval $[\alpha, \beta]$ of natural numbers and a C^1-neighborhood $\mathcal{U}_2$ of f, such that for every $g \in \mathcal{U}_2$, the set of indices of hyperbolic periodic points contained in $H(p_g, g)$ is $[\alpha, \beta]$. Also, all periodic points of the same index contained in $H(p)$ are homoclinically related.*

6. *If a homoclinic class $H(p)$ contains periodic orbits with different indices, then f can be C^1 approximated by diffeomorphisms having a heterodimensional cycle.*

7. *If a homoclinic class $H(p)$ is Lyapunov stable, then there is a C^1 neighborhood $\mathcal{U}_3$ of f, such that for any $g \in \mathcal{U}_3 \cap \mathcal{R}$, the homoclinic class $H(p_g, g)$ is also Lyapunov stable.*

8. *Consider a periodic point p. If for any C^1-neighborhood $\mathcal{U}$ of f, there is a diffeomorphism $g \in \mathcal{U}$ having a heterodimensional cycle associated with p_g and some periodic point q, then the homoclinic class $H(p)$ contains a periodic point whose index equals* $\mathrm{Ind}(q)$.

9. *Consider an invariant compact set K which is a Hausdorff limit set of a sequence of periodic orbits O_n. If for any neighborhood $\mathcal{U}$ of f and any $N > 0$, there is $g \in \mathcal{U}$ and $n > N$, such that O_n is a periodic orbit of g of index i, then K is the Hausdorff limit set of a sequence of periodic orbits of f of index i.*

10. *Consider a non-trivial homoclinic class $H(p)$ of f. For any $\varepsilon > 0$, and for any $i = 1, 2, \cdots, d$, the set*

$$\{q \in Per(f) \text{: } q \text{ has simple spectrum, and } |\chi_i(q) - \chi_i(p)| < \varepsilon\}$$

is dense in $H(p)$.

Chapter 3

Multi-connecting perturbations

In this chapter, we give several propositions allowing to connect several (even infinitely many) orbits at the same time. The proof of each proposition applies the connecting lemma several times. We point out here that, Propositions 3.1, 3.5 and 3.6 are applied to prove Theorem A in Chapter 4, and Proposition 3.3 is applied to prove Theorem C in Chapter 6.

3.1 Statement of the propositions

Propositions 3.1 and 3.3 aim to obtain periodic orbits that spend certain long time close to some invariant compact sets. Proposition 3.1 tells that for any hyperbolic periodic orbit $\mathrm{Orb}(p)$ and any invariant compact set K of a diffeomorphism f linked by heteroclinic orbits, we can get a periodic orbit that spends a given proportion of time close to $\mathrm{Orb}(p)$ and K by arbitrarily C^1 small perturbation, see Figure 3.1 when p is a fixed point.

Proposition 3.1. *Let $f \in \mathrm{Diff}^1(M)$. Consider a hyperbolic periodic point p and an invariant compact set K, satisfying that all periodic point contained in K are hyperbolic and $p \notin K$. Assume moreover that there are two points x and y, satisfying that:*

- *$x \in W^u(p)$ and $\omega(x) \cap K \neq \emptyset$,*
- *$y \in W^s(p)$ and $\alpha(y) = K$.*

Then for any neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, any neighborhood U_p of $\mathrm{Orb}(p)$, and any neighborhood U_K of K, there are two integers l and n_0, such that, for any integer T_K,

1. *there is $h \in \mathcal{U}$ such that:*
   - *h coincides with f outside U_K,*
   - *the point y is on the positive orbit of x under h,*

- $\#(\mathrm{Orb}(x,h)\cap U_K)\geq T_K$ *and* $\#((\mathrm{Orb}(x,h)\setminus(U_K\cup U_p))\leq n_0$.

2. *for any* $m\in\mathbb{N}$, *there is* $h_m\in\mathcal{U}$ *such that:*
   - h_m *coincides with* h *on* $\mathrm{Orb}(p)$ *and outside* U_p,
   - h_m *has a periodic orbit* O, *satisfying* $O\setminus U_p=(\mathrm{Orb}(x,h))\setminus U_p$, *and* $\#(O\cap U_p)\in\{l+m\tau,l+m\tau+1,\cdots,l+(m+1)\tau-1\}$.

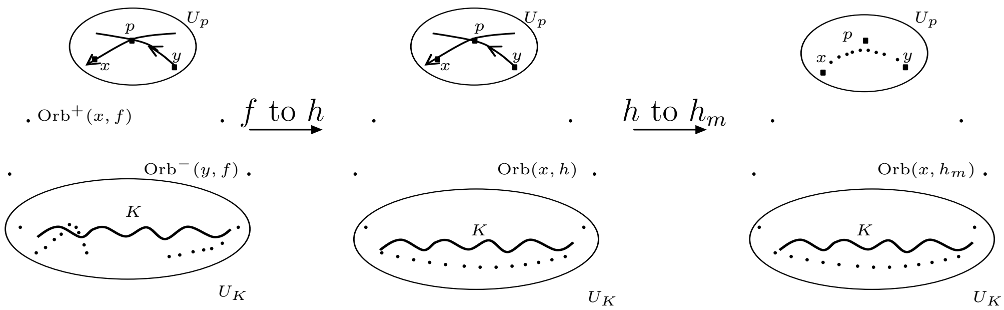

Figure 3.1: Periodic orbits shadows a set and a hyperbolic fixed point

Remark 3.2. *It is obvious that, in the settings of the proposition, if we change "$\omega(x)\cap K\neq\emptyset$ and $\alpha(y)=K$" to "$\alpha(y)\cap K\neq\emptyset$ and $\omega(x)=K$", the conclusion still holds. In the proof of Theorem A, we will use the assumption that "$\alpha(y)\cap K\neq\emptyset$ and $\omega(x)=K$".*

Proposition 3.3 gives periodic orbits that spend most of the time close to an invariant compact set and visit a small neighborhood of a point, see Figure 3.2 when p is a fixed point.

Proposition 3.3. *For generic $f\in\mathrm{Diff}^1(M)$, assume K is a chain transitive set of f and $x\in K$. If $x\notin\alpha(x)$, then for any C^1-neighborhood $\mathcal{U}$ of f, any neighborhood U of $\alpha(x)$, any neighborhood U_x of x and any neighborhood W of K, there is an integer $L\in\mathbb{N}$, with the following property. For any integer $m\in\mathbb{N}$, there is a diffeomorphism $g\in\mathcal{U}$ with a periodic point $p\in U_x$ whose orbit is contained in W, satisfying that:*

- $\#(\mathrm{Orb}(p,g)\cap U)\geq m$,
- $\#(\mathrm{Orb}(p,g)\setminus U)\leq L$.

Remark 3.4. (1) *Clearly, if we replace $\alpha(x)$ by $\omega(x)$ in the hypothesis, the conclusions are still valid. We point out here that, in the proof of Theorem C,*

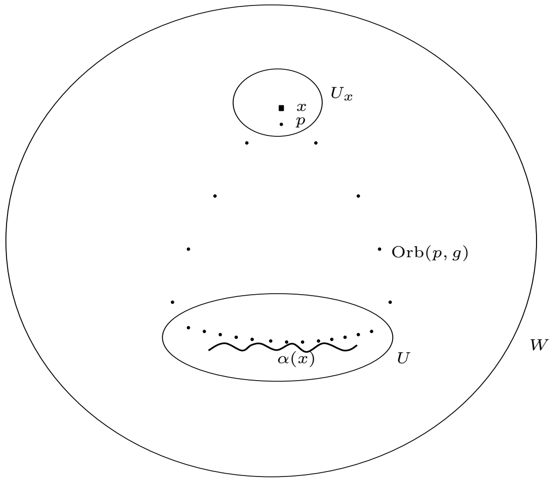

Figure 3.2: Periodic orbits travel close to a set and a point

we use the assumption that $x \notin \omega(x)$. But to simplify the notations, we prove Proposition 3.3 under the assumption $x \notin \alpha(x)$.

(2) *For the proof of Theorem C, we only have to consider the case where K contains no periodic point. But to give a general statement of Proposition 3.3, we will also prove the case when K contains periodic points.*

Proposition 3.5 and 3.6 are in some sense doing an asymptotic connecting process from a point to an invariant compact set. Proposition 3.5 tells that if a point on the unstable manifold of a periodic orbit satisfies that its positive limit set intersects an invariant compact set, then we can make its positive limit set contained in this invariant compact set by a small perturbation. Moreover, the perturbation will not change certain pieces of orbit. In fact, we can get the first property directly by the conclusions of [32], but the second property is not a direct consequence.

Proposition 3.5 (A modified case of Proposition 10 in [32])**.** *Let $f \in \mathrm{Diff}^1(M)$. Consider an invariant compact set K which contains no non-hyperbolic periodic point and a point $x \in M$ with $\alpha(x) \subset K$. Assume p is a hyperbolic periodic point satisfying that $p \notin K$ and $\overline{W^u(p)} \cap K \neq \emptyset$.*
Then for any neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, there is a diffeomorphism $g \in \mathcal{U}$, satisfying the following properties:

1. *g coincides with f on $\mathrm{Orb}(p) \cup K$;*

2. *g and Dg coincides with f and Df respectively on $\mathrm{Orb}^-(x)$, hence $\alpha(x, g) \subset K$;*

3. there is a point $y \in W^u(p, g)$, such that $\omega(y, g) \subset K$.

In the assumptions of Propositions 3.1 and 3.5, the point and invariant compact sets are linked by true orbits. However, Proposition 3.6 deals with the case that they are linked by pseudo-orbits which is more complicated. We use the technics of [15, 32].

Proposition 3.6. *Assume f_0 is a diffeomorphism in $\mathrm{Diff}^1(M)$. For any neighborhood $\mathcal{U}$ of f_0 in $\mathrm{Diff}^1(M)$, there are a smaller neighborhood $\mathcal{U}'$ of f_0 with $\overline{\mathcal{U}'} \subset \mathcal{U}$ and an integer T, with the following properties.*
For any diffeomorphism $f \in \mathcal{U}'$, considering an invariant compact set K, a positively invariant compact set X and a point $z \in X$, suppose that the following conditions are satisfied:

- *all periodic orbits contained in K are hyperbolic,*
- *all periodic orbits contained in X with period less than T are hyperbolic,*
- *for any $\varepsilon > 0$, there is an ε-pseudo-orbit contained in X connecting z to K,*

then for any neighborhood U of $X \setminus K$ and for any $\gamma > 0$, there is a diffeomorphism $g \in \mathcal{U}$, such that: $g|_{M\setminus U} = f|_{M\setminus U}$ and $\omega(z, g) \subset K$. Moreover, the C^0 distance between g and f is smaller than γ.

Remark 3.7. (1) *Proposition 3.5 is not a direct corollary of Proposition 3.6, because we wish to keep the negative orbit of a point that accumulates to the invariant compact set unchanged after perturbation in Proposition 3.5.*

(2) *In Proposition 3.6, we can see that $X \cap K \neq \emptyset$. Thus $X \setminus K$ is not a compact set and we have that $\overline{U} \cap K \neq \emptyset$, where U is the neighborhood of $X \setminus K$.*

(3) *In the proof, we will show that the perturbation neighborhoods can be chosen to have very small diameters, hence the C^0 distance between g and f can be small.*

The following sections in this chapter give the proofs of the above propositions.

3.2 Periodic orbits shadowing a periodic orbit and a set: proof of Proposition 3.1

In this section, we give the proof of Proposition 3.1. We consider the periodic point p and the invariant compact set K satisfying that $p \notin K$ and K contains no non-hyperbolic periodic point. Recall that the point $x \in W^u(p)$ satisfies $\omega(x) \cap K \neq \emptyset$ and the point $y \in W^s(p)$ satisfies $\alpha(y) = K$.

Taking a smaller neighborhood if necessary, we can assume that the element of $\mathcal{U}$ is of the form $f \circ \phi$ with $\phi \in \mathcal{V}$, where $\mathcal{V}$ is a C^1 neighborhood of Id and satisfies the property (F) in Definition 2.24. By Theorem 2.25, there is an integer N associated to the neighborhood $\mathcal{U}$. By the Lemma 2.27, there are two numbers $\theta > 1$ and $r_0 > 0$ associated to $\mathcal{U}$.

It is easy to see that we only need to prove the proposition for U_p and U_K small. More precisely, we assume that $U_p \cap U_K = \emptyset$ and $(\mathrm{Orb}^-(x) \cup \mathrm{Orb}^+(y)) \cap U_K = \emptyset$. Moreover, by the hyperbolicity of periodic orbits in K, we assume that there are no periodic points with period less than or equal to N contained in $U_K \setminus K$.

To simplify the notation, we assume that p is a hyperbolic fixed point of f, and the proof of the general case is similar. The only difference is that, in item 2 of the conclusion, the condition $\#(O \cap U_p) \in \{l + m\tau, l + m\tau + 1, \cdots, l + (m+1)\tau - 1\}$ should be $\#(O \cap U_p) = l + m$. In the general case, the number $\#(O \cap U_p)$ cannot be chosen arbitrarily, but we can make sure it is contained in an interval whose length is the period of p.

Now we fix the neighborhoods $\mathcal{U}$, U_p and U_K, and the numbers N, θ and r_0.

3.2.1 The choice of n_0, the point z_1 and the perturbation domain at z_1.

Recall that the point $x \in W^u(p)$ satisfies $\omega(x) \cap K \neq \emptyset$.

Lemma 3.8. *There is a point $z_1 \in U_K \setminus K$, such that:*

- *for any neighborhood V_{z_1} of z_1, there is $n \geq 1$ such that $f^n(x) \in V_{z_1}$;*
- *$z_1 \prec_{U_K} K$ and $\mathrm{Orb}^+(z_1) \subset U_K$.*

Proof. The proof is similar to that of Lemma 2.14. We take a smaller open neighborhood V of K such that $\overline{V} \subset U_K$. Since $\omega(x) \cap K \neq \emptyset$, then for any $k \geq 1$, there is $n_k \geq 1$, such that $f^{n_k}(x) \in B(K, \frac{1}{k})$. Take the smallest integer m_k, such that the piece of orbit $(f^{m_k}(x), f^{m_k+1}(x), \cdots, f^{n_k}(x))$ is contained in V. Taking a convergent subsequence if necessary, we assume that the sequence $\{f^{m_k}(x)\}_{k \geq 1}$ converges to a point $z_1 \in \overline{V} \subset U_K$ and the sequence $\{f^{n_k}(x)\}_{k \geq 1}$ converges to a point $z_2 \in K$. Then we have that $z_1 \prec_{U_K} z_2$, and the pieces of orbit that connects the neighborhoods of z_1 and z_2 are $(f^{m_k}(x), f^{m_k+1}(x), \cdots, f^{n_k}(x))_{k \geq 1}$. Since $z_2 \in K$, we have that $z_1 \prec_{U_K} K$. By the choice of m_k, we have that $f^{m_k-1}(x) \in M \setminus V$. Since $M \setminus V$ is compact, and $f^{-1}(z_1)$ is a limit point of the sequence $\{f^{m_k-1}(x)\}_{k \geq 1}$, we have that $f^{-1}(z_1) \in M \setminus V$. By the invariance of K, we have that $z_1 \notin K$ and $n_k - m_k$ goes to $+\infty$. Since $(f^{m_k}(x), f^{m_k+1}(x), \cdots, f^{n_k}(x))$ is contained in U_K and by the fact that $f^{m_k}(x)$ converges to z_1, we have that $\mathrm{Orb}^+(z_1) \subset \overline{V} \subset U_K$. Thus the second item is satisfied. The first item is a trivial fact by the choice of z_1. □

By the assumption on U_K, we have that z_1 is not a periodic point with period less than or equal to N. Also, since $y \in W^s(p)$ and $\mathrm{Orb}^+(z_1) \subset U_K$, we have $z_1 \notin \mathrm{Orb}(y)$. Moreover, by the fact that $\alpha(y) = K$, we know $z_1 \notin \overline{\mathrm{Orb}(y)}$. Then there are two neighborhoods $V_{z_1} \subset U_{z_1}$ of z_1 satisfying the conclusion of Theorem 2.25 for the triple $(f, \mathcal{U}, N)$, and also satisfying the following conditions:

- $U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1}) \subset U_K \setminus K$;
- $(U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})) \cap \mathrm{Orb}(y) = \emptyset$.

By Lemma 3.8, there is $n_1 \in \mathbb{N}$ such that $f^{n_1}(x) \in V_{z_1}$. Moreover, since $\alpha(y) = K$, there is $n_2 \in \mathbb{N}$, such that, for any $n \geq n_2$, we have $f^{-n}(y) \in U_K$. Let $n_0 = n_1 + n_2$.

3.2.2 The choices of points and perturbation domains in K and to get h.

Take any integer T_K. By Lemma 3.8, we have that $z_1 \prec_{U_K} K$, that is to say, there is a point $z_2 \in K$ such that $z_1 \prec_{U_K} z_2$. Now we consider two cases, depending on whether there is such a point z_2 that is not a periodic point with period less than or equal to N.

The non-periodic case

Assume that there is a point $z_2 \in K$ which is not a periodic point with period less than or equal to N, and $z_1 \prec_{U_K} z_2$. Then there are two neighborhoods $V_{z_2} \subset U_{z_2}$ of z_2 satisfying the conclusion of Theorem 2.25 for the triple $(f, \mathcal{U}, N)$ and also satisfying the following conditions:

- $U_{z_2} \cup f(U_{z_2}) \cup \cdots \cup f^N(U_{z_2}) \subset U_K$;
- $(U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})) \cap (U_{z_2} \cup f(U_{z_2}) \cup \cdots \cup f^N(U_{z_2})) = \emptyset$;
- $f^n(x) \notin U_{z_2} \cup f(U_{z_2}) \cup \cdots \cup f^N(U_{z_2})$, for any $-\infty < n \leq n_1$;
- $f^{-n}(y) \notin U_{z_2} \cup f(U_{z_2}) \cup \cdots \cup f^N(U_{z_2})$, for any $n \leq n_2 + T_K$.

Then there is $n_3 > n_2 + T_K$, such that $f^{-n_3}(y) \in V_{z_2}$. Since we have the fact that $z_1 \prec_{U_K} z_2$, there is a piece of orbit $(w, f(w), \cdots, f^k(w))$ contained in U_K, such that $w \in V_{z_1}$ and $f^k(w) \in V_{z_2}$. Moreover, since $U_{z_1} \cap \mathrm{Orb}(y) = \emptyset$ and $w \in V_{z_1} \subset U_{z_1}$, we have that $w \notin \mathrm{Orb}(y)$.

Perturbations to get h in the non-periodic case. Now we do the perturbations step by step to get the conclusion.

Step 1. From the choice of points and neighborhoods above, we can see that the point x has a positive iterate $f^{n_1}(x) \in V_{z_1}$ and the point $f^k(w)$ has a negative iterate $w \in V_{z_1}$. Then by Theorem 2.25, there is a diffeomorphism $f_1 \in \mathcal{U}$, such that f_1 coincides with f outside $U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})$ and $f^k(w)$ is on the positive orbit of x under f_1.

Step 2. For the diffeomorphism f_1, the point x has a positive iterate $f^k(w) \in V_{z_2}$, and the point y has a negative iterate $f^{-n_3}(y) \in V_{z_2}$. Since f_1 coincides with f outside $U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})$ and $(U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})) \cap (U_{z_2} \cup f(U_{z_2}) \cup \cdots \cup f^N(U_{z_2})) = \emptyset$, then by Theorem 2.25, there is a diffeomorphism $h \in \mathcal{U}$, such that y is on the positive orbit of x under h and h coincides with f_1 outside $U_{z_2} \cup f(U_{z_2}) \cup \cdots \cup f^N(U_{z_2})$.

By the constructions above, h coincides with f outside $(U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})) \cup (U_{z_2} \cup f(U_{z_2}) \cup \cdots \cup f^N(U_{z_2}))$. Hence h coincides with f on $\mathrm{Orb}(p) \cup \mathrm{Orb}^-(x) \cup \mathrm{Orb}^+(y)$ and outside U_K. Moreover, $\#(\mathrm{Orb}(x,h) \cap U_K) \geq n_3 - n_2 \geq T_K$ and $\#(\mathrm{Orb}(x,h) \setminus (U_K \cup U_p)) \leq n_1 + n_2 \leq n_0$.

The periodic case

Assume that any point $z_2 \in K$ satisfying $z_1 \prec_{U_K} z_2$ is a periodic point with period less than or equal to N. We take such a point $q \in K$. In this case, we cannot use Theorem 2.25 at the point q since its period is small but we can do perturbations at the stable and unstable manifolds of q since it is hyperbolic. To simplify the proof, we assume that q is a hyperbolic fixed point of f, but the general case is identical. We take a neighborhood U_q of q such that $\overline{U_q} \subset U_K \setminus (U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1}))$, and such that for any point w satisfying $\mathrm{Orb}^+(w) \subset U_q$ (rep. $\mathrm{Orb}^-(w) \subset U_q$), we have $w \in W^s(q)$ (resp. $w \in W^u(q)$).

Since $z_1 \prec_{U_K} q$, by a similar argument as in Lemma 3.8, we can get that, there is a point $x' \in U_q$, such that $z_1 \prec_{U_K} x'$ and $\mathrm{Orb}^+(x) \subset U_q$. By the choice of U_q, we have that $x' \in W^s(q)$ and $x' \notin U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})$. Notice that, since x' is not a periodic point, we have $x' \notin K = \alpha(y)$.

Since $q \in \alpha(y)$, there is $y' \in W^u(q) \cap U_q$, such that for any neighborhood U of y', there is an integer $n \geq 1$, such that $f^{-n}(y) \in U$. (In fact, if $\alpha(y) = \{q\}$, we can choose y' to be a negative iterate of y. If $\{q\} \subsetneqq \alpha(y)$, we can choose y' to be contained in $\alpha(y) \cap W^u(q)$).

By the λ-Lemma, there are two neighborhoods $W_{x'}$ and $W_{y'}$ of x' and y' respectively such that, for any two smaller neighborhoods $W'_{x'} \subset W_{x'}$ and $W'_{y'} \subset W_{y'}$ of x' and y' respectively, there are a piece of orbit $(z', f(z'), \cdots, f^t(z'))$ contained in U_q, such that $z' \in V_{x'}$, $f^t(z') \in f^{-1}(W_{y'})$, $f^i(z') \notin U_{x'}$ for any $i \in \{1, 2, \cdots, t\}$ and $t \geq T_K$.

Now we construct the perturbation domains at the points x' and y' respectively. Recall that $\theta > 1$ and $r_0 > 0$ are the two constants obtained by Lemma 2.27 associated to $\mathcal{U}$ and f.

Perturbation domain at x' We can take two neighborhoods $V_{x'} \subset U_{x'}$ of x' that satisfy the conclusions of Theorem 2.25 for the triple $(f, \mathcal{U}, N)$, and also satisfy that

- $U_{x'} \subset W_{x'}$;
- $U_{x'} \cup f(U_{x'}) \cup \cdots, \cup f^N(U_{x'}) \subset U_q$;
- $f^n(x) \notin U_{x'} \cup f(U_{x'}) \cup \cdots \cup f^N(U_{x'})$, for any $-\infty < n \leq n_1$;
- $U_{x'} \cap \mathrm{Orb}(y) = \emptyset$ and $q \notin U_{x'}$.

Then, there is a piece of orbit $(w', \cdots, f^{k'}(w'))$ contained in U_K, such that $w' \in V_{z_1}$ and $f^{k'}(w') \in V_{x'}$. Moreover, $w' \notin \mathrm{Orb}(y)$.

Perturbation domain at y' We take a number $r' < r_0$ small enough, such that: if we take the neighborhood $U_{y'} = f(B(f^{-1}(y'), \theta r'))$ of y', then the following properties are satisfied:

- $U_{y'} \subset W_{f^{-1}(y')}$;
- $U_{y'} \cup f^{-1}(U_{y'}) \subset U_q \setminus (\{q\} \cup U_{x'} \cup f(U_{x'}) \cup \cdots, \cup f^N(U_{x'}))$;
- $U_{y'} \cap f^{-1}(U_{y'}) = \emptyset$;
- $f^n(x) \notin U_{y'} \cup f^{-1}(U_{y'})$, for any $-\infty < n \leq n_1$;
- $\{w', \cdots, f^{k'}(w')\} \cap (U_{y'} \cup f^{-1}(U_{y'})) = \emptyset$.

Then by the choice of $U_{x'}$ and $U_{y'}$, there is a piece of orbit $(z', f(z'), \cdots, f^{n_4}(z'))$ contained in U_q, such that $z' \in V_{x'}$, $f^{n_4}(z') \in B(f^{-1}(y'), r')$, $f^i(z') \notin U_{x'}$ for any $i \in \{1, 2, \cdots, n_4\}$ and $n_4 \geq T_K$. By the choice of y', there is a negative iterate $f^{-n_5}(y)$ of y contained in $B(f^{-1}(y'), r')$.

Perturbations to get h in the periodic case. From the above constructions, we can see that the perturbation domains are pairwise disjoint and contained in U_K, and the pieces of orbits that connects two perturbation domains are pairwise disjoint and disjoint from the other perturbation domains. Then we can do the perturbations step by step as in *Case 1.*

Step 1. By Lemma 2.27, there is $f_1 \in \mathcal{U}$, such that, f_1 coincides with f outside $f^{-1}(U_{y'})$ and $f_1(f^{n_4}(z')) = f^{-n_5+1}(y)$. Since $f^n(x) \notin U_{y'} \cup f^{-1}(U_{y'})$, for any $-\infty < n \leq n_1$, we have that f_1 coincides with f on $\{f^n(x)\}_{-\infty<n\leq n_1}$.

Step 2. For the diffeomorphism f_1, the point y has a negative iterate $z' \in V_{x'}$, and the point w' has a positive iterate $f^{k'}(w') \in V_{x'}$. Then by

Theorem 2.25, there is $f_2 \in \mathcal{U}$, such that f_2 coincides with f_1 outside $U_{x'} \cup f(U_{x'}) \cup \cdots \cup f^N(U_{x'})$, and w' is on the negative orbit of y under f_2. $f^n(x) \notin U_{x'} \cup f(U_{x'}) \cup \cdots \cup f^N(U_{x'})$, for any $-\infty < n \leq n_1$, we have that f_1 coincides with f on $\{f^n(x)\}_{-\infty<n\leq n_1}$.

Step 3. For the diffeomorphism f_2, the point y has a negative iterate w' in V_{z_1} and the point x has a positive iterate $f^{n_1}(x) \in V_{z_1}$. By Theorem 2.25, there is $h \in \mathcal{U}$, such that, h coincides with f_2 outside $U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})$ and y on the positive orbit of x under h.

By the constructions above, the diffeomorphism h coincides with f outside $(U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})) \cup (U_{x'} \cup f(U_{x'}) \cup \cdots \cup f^N(U_{x'})) \cup f^{-1}(U_{y'})$. Hence h coincides with f on $\mathrm{Orb}(p) \cup \mathrm{Orb}^-(x) \cup \mathrm{Orb}^+(y)$ and outside U_K. Moreover, $\#(\mathrm{Orb}(x,h) \cap U_K) \geq n_4 \geq T_K$ and $\#(\mathrm{Orb}(x,h) \setminus (U_K \cup U_p)) \leq n_1 + n_2 \leq n_0$.

Remark 3.9. *We point out that, in the construction, each of the perturbations supports are pairwise disjoint with each other. Since $\mathcal{U} = f \circ \mathcal{V}$, where $\mathcal{V}$ is a C^1 neighborhood of Id and satisfies the property (F) in Definition 2.24, the perturbations f_1, f_2 and h are still contained in $\mathcal{U}$.*

3.2.3 The choice of l and the perturbation domains at x and y, and to get h_m.

From the constructions in section 3.2.2, we get the diffeomorphism h that satisfies the first item of Proposition 3.1. In this section, we do the perturbations to get the diffeomorphism h_m.

Assume $h^t(x) = y$. By replacing x and y to a negative or positive iteration, we assume that $x, y \in U_p$ and $\mathrm{Orb}^-(x,h) \cup \mathrm{Orb}^+(y,h) \subset U_p$. Assume that $\#(\{h^n(x)\}_{1\leq n\leq t} \cap U_p) = m_0$. We take a number $r < r_0$ small enough, such that, if we take the neighborhood $U_x = h(B(h^{-1}(x), \theta r))$ of x and the neighborhood $U_y = B(y, \theta r)$ of y, then, the four sets U_x, $h^{-1}(U_x)$, U_y and $h(U_y)$ are contained in U_p and pairwise disjoint from each other and disjoint with $\{h^n(x)\}_{1\leq n\leq t}$. By the λ-Lemma, there is $l_0 \in \mathbb{N}$, such that, for any $m \geq 1$, there is a piece of orbit $(h(z), h^2(z), \cdots, h^{l_0+m-1}(z))$ contained in U_p, such that $h(z) \in B(y, r)$, $h^{l_0+m-1}(z) \in B(h^{-1}(x), r)$ and $h^i(z) \notin U_y \cup h^{-1}(U_x)$ for any $i = 2, 3, \cdots, l_0 + m - 2$. Let $l = l_0 + m_0$.

By Lemma 2.27 and the disjointness of U_y, $f^{-1}(U_x)$ and U_K, there is $h_m \in \mathcal{U}$, such that, h_m coincides with h outside $U_y \cup f^{-1}(U_x)$, and $h_m(y) = h^2(z)$, $h_m(f^{l_0+m-1}(z)) = h_m^{l_0+m-1}(y) = x$. Hence h_m coincides with h on $\mathrm{Orb}(p)$ and outside U_p. Moreover, the point x is a periodic point of h_m, and putting $O = \mathrm{Orb}(x, h_m)$, we have that $O \setminus U_p = \mathrm{Orb}(x,h) \setminus U_p$, and $\#(O \cap U_p) = l_0 + m_0 + m = l + m$.

This finishes the proof of Proposition 3.1.

Remark 3.10. *We point out here that, in the periodic case, we cannot do the same perturbations at x' to connect $f^{k'}(w')$ to $f(z')$ just as at the points y' to* *connect $f^{n_4}(z')$ and $f^{-n_5+1}(y)$. Because the piece of orbit $(w', \cdots, f^{k'}(w'))$ that connects the neighborhoods V_{z_1} and $V_{x'}$ of z_1 and x' respectively may enter into the neighborhood $U_{x'}$ many times before $f^{k'}(w')$. Thus if we use the basic perturbation lemma to connect $f^{k'}(w')$ to $f(z')$, the piece of orbit $(w', \cdots, f^{k'}(w'))$ may be modified and it is not clear if the negative orbit of y can intersect V_{z_1} after such perturbation. Thus we can not get a periodic orbit.*

3.3 Connecting a set and a point by periodic orbits: proof of Proposition 3.3

In this section, we prove Proposition 3.3.

Consider a diffeomorphism f that satisfies the properties stated in Theorem 2.46, a chain-transitive set K of f, a point $x \in K$ satisfying $x \notin \alpha(x)$ (hence x is not a periodic point) and a C^1-neighborhood $\mathcal{U}$ of f. By taking a smaller neighborhood if necessary, we assume that the elements of $\mathcal{U}$ are of the form $f \circ \phi$ with $\phi \in \mathcal{V}$, such that $\mathcal{V}$ is a C^1-neighborhood of Id which satisfies the property (F). For the C^1-neighborhood $\mathcal{U}$ of f, there is an integer N given by Theorem 2.25.

We fix the triple $(f, \mathcal{U}, N)$, and fix the three neighborhoods U_x of x, U of $\alpha(x)$ and W of K from now on. We consider two cases.

- **The non-periodic case:** there is a point $z \in \alpha(x)$ such that z is not a periodic point of f with period less than or equal to N.
- **The periodic case:** any point contained in $\alpha(x)$ is a periodic point of f with period less than or equal to N, which in particular implies that $\alpha(x)$ is a finite set.

3.3.1 The non-periodic case.

We construct three perturbation neighborhoods at three points, and choose segments of orbits that connect them one by one and then we use the connecting lemma to get a periodic orbit by perturbations. We point out here that the perturbation neighborhoods are pairwise disjoint and the segments of orbits are also pairwise disjoint. Moreover, any perturbation neighborhood is disjoint with the segment of orbit that connects the two other perturbation neighborhoods.

Choice of points and connecting orbits

The perturbation neighborhoods at x. We take two small neighborhoods $V'_x \subset V_x \subset U_x$ of x, such that the following properties are satisfied:

- $(\bigcup_{i=0}^{N} f^i(\overline{V_x})) \cap \alpha(x) = \emptyset$,

- $\bigcup_{i=0}^{N} f^i(V_x) \subset W$,
- $V'_x \subset V_x$ satisfy Theorem 2.25 for the triple $(f, \mathcal{U}, N)$.

The point y. Since all periodic points of f are hyperbolic, then there are only finitely many periodic points with period less than or equal to N. Hence there is a small neighborhood $V \subset U$ of $\alpha(x)$, such that

- $V \subset W$ and $\overline{V} \cap (\bigcup_{i=0}^{N} f^i(\overline{V_x})) = \emptyset$,
- there is no periodic point with period less than or equal to N in $\overline{V} \setminus K$.

Since K is a chain transitive set and $x \in K$, we have that $x \dashv_K \alpha(x)$. By item 3 of Theorem 2.46, we have that $x \prec_W \alpha(x)$. By Lemma 2.14, there is a point $y \in V \setminus \alpha(x)$, such that $x \prec_W y \prec_V \alpha(x)$, and $\mathrm{Orb}^+(y) \subset V$. Since $x \notin V$, we have that $y \notin \overline{\mathrm{Orb}^-(x)}$. Moreover, there is an integer n_0, such that, for any $i \geq n_0$, $f^{-i}(x) \subset V$.

The perturbation neighborhoods at y, the connecting orbit from x to y and the number L. By the choice the neighborhood V, the point y is not a periodic point with period less than or equal to N. By the facts that $y \notin \overline{\mathrm{Orb}^-(x)}$ and $\mathrm{Orb}^+(y) \subset V$, there are two small neighborhoods $V'_y \subset V_y$ of y, such that the following properties are satisfied:

- $\bigcup_{i=0}^{N} f^i(V_y) \subset V$, which implies that $(\bigcup_{i=0}^{N} f^i(\overline{V_y})) \cap (\bigcup_{i=0}^{N} f^i(\overline{V_x})) = \emptyset$,
- $\overline{\mathrm{Orb}^-(x)} \cap (\bigcup_{i=0}^{N} f^i(\overline{V_y})) = \emptyset$, which implies that $(\bigcup_{i=0}^{N} f^i(\overline{V_y})) \cap \alpha(x) = \emptyset$,
- $V'_y \subset V_y$ satisfy Theorem 2.25 for the triple $(f, \mathcal{U}, N)$.

Since $x \prec_W y$, there is a piece of orbit segment $\{w_1, f(w_1), \cdots, f^{n_1}(w_1)\} \subset W$, such that $w_1 \in V'_x$ and $f^{n_1}(w_1) \in V'_y$. By the choice of V_y, one can see that $w_1 \notin \mathrm{Orb}^-(x)$. Take $L = n_0 + n_1 + N$. Then we take an integer $m \in \mathbb{N}$.

The point z and the perturbation neighborhoods at z. Now take $z \in \alpha(x)$ such that z is not a periodic point of f with period less than or equal to N. Since $\alpha(x)$ is a chain transitive set and $y \prec_V \alpha(x)$, by item 3 of Theorem 2.46, we know that $y \prec_V z$. By the fact that $z \in \alpha(x)$ and $(\bigcup_{i=0}^{N} f^i(\overline{V_y})) \cap \alpha(x) = \emptyset$, we can take two neighborhoods $V'_z \subset V_z$ of z, such that the following properties are satisfied:

- $\bigcup_{i=0}^{N} f^i(V_z) \subset V$, which implies that $(\bigcup_{i=0}^{N} f^i(\overline{V_z})) \cap (\bigcup_{i=0}^{N} f^i(\overline{V_x})) = \emptyset$,
- $(\bigcup_{i=0}^{N} f^i(\overline{V_z})) \cap (\bigcup_{i=0}^{N} f^i(\overline{V_y})) = \emptyset$,
- $\{w_1, f(w_1), \cdots, f^{n_1}(w_1)\} \cap (\bigcup_{i=0}^{N} f^i(\overline{V_z})) = \emptyset$,

- $f^{-i}(x) \notin \bigcup_{i=0}^{N} f^i(V_z)$, for any $0 \le i \le n_0 + m$,
- $V_z' \subset V_z$ satisfy Theorem 2.25 for the triple $(f, \mathcal{U}, N)$.

The connecting orbits from y to z and from z to x. Since $y \prec_V z$, there is a piece of orbit segment $\{w_2, f(w_2), \cdots, f^{n_2}(w_2)\} \subset V$, such that $w_2 \in V_y'$ and $f^{n_2}(w_2) \in V_z'$. By the choice of V_y, we have that $w_2 \notin \mathrm{Orb}^-(x)$. By the choice of the neighborhood V, we have that $\{w_2, f(w_2), \cdots, f^{n_2}(w_2)\} \cap (\bigcup_{i=0}^{N} f^i(\overline{V_x})) = \emptyset$.

Since $z \in \alpha(x)$, there is n_3, such that $f^{-n_3}(x) \in V_z'$. Since $V_y \cap \mathrm{Orb}^-(x) = \emptyset$, we have that $\{f^{-n_3}(x), f^{-n_3+1}(x), \cdots, x\} \cap (\bigcup_{i=0}^{N} f^i(\overline{V_y})) = \emptyset$. Moreover, by the choice of V_z, we have that $n_3 > n_0 + m$.

To sum up, we have obtained three pairwise disjoint perturbation neighborhoods $\bigcup_{i=0}^{N} f^i(V_x)$, $\bigcup_{i=0}^{N} f^i(V_y)$ and $\bigcup_{i=0}^{N} f^i(V_z)$, and three pairwise disjoint pieces of orbit segment $\{w_1,\ f(w_1),\ \cdots,\ f^{n_1}(w_1)\}$, $\{w_2,\ f(w_2),\ \cdots,\ f^{n_2}(w_2)\}$ and $\{f^{-n_3}(x),\ f^{-n_3+1}(x),\ \cdots,\ x\}$ that connect the perturbation neighborhoods one by one. And all these perturbation neighborhoods and pieces of orbit segments are contained in the neighborhood W of K, and each perturbation neighborhood is disjoint with the piece of orbit segment that connects the other two perturbation neighborhoods. Moreover, we have that $n_3 > n_0 + m > m$.

The connecting process

Now, by Theorem 2.25, we will do perturbations of f in $\mathcal{U}$ on the three pairwise disjoint neighborhoods $\bigcup_{i=0}^{N} f^i(V_x)$, $\bigcup_{i=0}^{N} f^i(V_y)$ and $\bigcup_{i=0}^{N} f^i(V_z)$. By Remarque 4.3 of [15] and the Property (F), one can get a diffeomorphism in $\mathcal{U}$ with the composition of the three perturbations.

The perturbation at x. The point $f^{-n_3}(x)$ has a positive iterate $x \in V_x'$ and the point $f^{n_1}(w_1)$ has a negative iterate $w_1 \in V_x'$. By Theorem 2.25, there is a diffeomorphism $f_1 \in \mathcal{U}$, such that:

- f_1 coincides with f outside $\bigcup_{i=0}^{N} f^i(V_x)$,
- the point $f^{n_1}(w_1)$ is on the positive orbit of $f^{-n_3}(x)$ under f_1.

Moreover, the piece of orbit segment $\{f^{-n_3}(x),\ f_1(f^{-n_3}(x)),\ \cdots,\ f^{n_1}(w_1)\}$ under f_1 satisfies the following properties:

- it is contained in $\bigcup_{i=0}^{n_3}\{f^{-i}(x)\} \cup (\bigcup_{i=0}^{N-1} f^i(V_x)) \cup \bigcup_{i=0}^{n_1}\{f^i(w_1)\}$,
- it intersect V_x and has at most $n_0 + n_1$ points outside V, where $n_0 + n_1 < L$,

- it contains the piece of orbit segment $\{f^{-n_0}(x), f^{-n_0-1}(x), \cdots, f^{-n_0-m}(x)\}$ under f.

We take a point $p \in V_x \cap \{f^{-n_3}(x), f_1(f^{-n_3}(x)), \cdots f^{n_1}(w_1)\}$, then the point $f^{n_1}(w_1)$ is on the positive orbit of p and $f^{-n_3}(x)$ is on the negative orbit of p under f_1.

The perturbation at y. By the above construction, f_1 coincides with f in $\bigcup_{i=0}^{N} f^i(V_y)$. Hence the piece of orbit $\{w_2, f(w_2), \cdots, f^{n_2}(w_2)\}$ is not modified. Under the iterate of f_1, the point p has a positive iterate $f^{n_1}(w_1) \in V'_y$ and the point $f^{n_2}(w_2)$ has a negative iterate $w_2 \in V'_y$. By Theorem 2.25, there is $f_2 \in \mathcal{U}$, such that:

- f_2 coincides with f_1 outside $\bigcup_{i=0}^{N} f^i(V_y)$, hence f_2 coincides with f outside $(\bigcup_{i=0}^{N} f^i(V_x)) \cup (\bigcup_{i=0}^{N} f^i(V_y))$,
- $f^{n_2}(w_2)$ is on the positive iterate of p under f_2, and $f^{-n_3}(x)$ is on the negative of p under f_2,
- the piece of orbit segment $\{f^{-n_3}(x),\ f_2(f^{-n_3}(x)),\ \cdots,\ p,\ f_2(p),\ \cdots,\ f^{n_2}(w_2)\}$ under f_2 has at most $L = n_0 + n_1 + N$ points outside V,
- the piece of orbit segment $\{f^{-n_0}(x),\ f^{-n_0-1}(x),\ \cdots,\ f^{-n_0-m}(x)\}$ under f is contained in the piece of orbit segment $\{f^{-n_3}(x),\ f_2(f^{-n_3}(x)),\ \cdots,\ p,\ f_2(p),\ \cdots,\ f^{n_2}(w_2)\}$ under f_2.

The perturbation at z. By the above constructions, f_2 coincides with f in $\bigcup_{i=0}^{N} f^i(V_z)$, and, under the iterate of f_2, the point p has a positive iterate $f^{n_2}(w_2) \in V'_z$ and a negative iterate $f^{-n_3}(x) \in V'_z$. By Theorem 2.25, there is $g \in \mathcal{U}$, such that:

- g coincides with f_2 outside $\bigcup_{i=0}^{N} f^i(V_z)$, hence g coincides with f outside $(\bigcup_{i=0}^{N} f^i(V_x)) \cup (\bigcup_{i=0}^{N} f^i(V_y)) \cup (\bigcup_{i=0}^{N} f^i(V_z))$,
- the point $p \in U_x$ is a periodic point of g,
- the piece of orbit segment $\{f^{-n_0}(x), f^{-n_0-1}(x), \cdots, f^{-n_0-m}(x)\}$ under f is contained in $\mathrm{Orb}(p, g)$, hence $\mathrm{Orb}(p, g)$ has at least m points contained in $V \subset U$,
- $\mathrm{Orb}(p, g)$ has at most $L = n_0 + n_1 + N$ points outside $V \subset U$.

This finishes the proof of Proposition 3.3 in the non-periodic case.

3.3.2 The periodic case.

In this case, any point contained in $\alpha(x)$ is a periodic point with period less than or equal to N. By the assumption that all periodic point of f is hyperbolic, we have that $\alpha(x)$ is a finite set. Since $\alpha(x)$ is chain transitive, this gives the following claim.

Claim 3.11. *In this case, $\alpha(x)$ is a hyperbolic periodic orbit* $\mathrm{Orb}(q)$*, and* $x \in W^u(\mathrm{Orb}(q))$.

Choice of points and connecting orbits

By Claim 3.11, we have that $x \in W^u(\mathrm{Orb}(q)) \setminus \{q\}$ for a hyperbolic periodic point q. To simplify the notations, we just assume that q is a hyperbolic fixed point of f, but the general case is identical. Now U is a neighborhood of $\{q\} = \alpha(x)$.

The perturbation neighborhoods at x. We take two small neighborhoods $V'_x \subset V_x \subset U_x$ of x, such that the following properties are satisfied:

- $q \notin \bigcup_{i=0}^{N} f^i(\overline{V_x})$,
- $\bigcup_{i=0}^{N} f^i(V_x) \subset W$,
- $V'_x \subset V_x$ satisfy Theorem 2.25 for the triple $(f, \mathcal{U}, N)$.

The neighborhood V and the point y. By the hyperbolicity of the fixed point q, there is a neighborhood V of q, such that, if the positive orbit of a point is contained in $\overline{V}$, then this point is in $W^s(q)$. Moreover, we can assume V is small such that $\overline{V} \cap (\bigcup_{i=0}^{N} f^i(\overline{V_x})) = \emptyset$ and $\overline{V} \subset U$. By the assumption, we have that $x \dashv_K q$, hence by item 3 of Theorem 2.46, we have that $x \prec_W q$. By Lemma 2.14, there is a point $y \in V \setminus q$, such that $x \prec_W y \prec_V q$, and $\mathrm{Orb}^+(y) \subset V$. Then we can see that $y \in W^s(q)$. Moreover, since $x \notin V$, we have that $y \notin \overline{\mathrm{Orb}^-(x)}$.

The perturbation neighborhoods at y, the connecting orbit from x to y and the number L. We take two neighborhoods $V'_y \subset V_y$ of y, such that the followings are satisfied:

- $\bigcup_{i=0}^{N} f^i(\overline{V_y}) \subset V \setminus \overline{\mathrm{Orb}^-(x)}$, which implies that:
$$(\bigcup_{i=0}^{N} f^i(\overline{V_y})) \cap (\bigcup_{i=0}^{N} f^i(\overline{V_x})) = \emptyset,$$
- $f^j(y) \notin \bigcup_{i=0}^{N} f^i(\overline{V_y})$, for any $j \geq N+1$,
- $V'_y \subset V_y$ satisfy Theorem 2.25 for the triple $(f, \mathcal{U}, N)$.

Since $x \prec_W y$, there is a piece of orbit segment $\{w_1, f(w_1), \cdots, f^{n_1}(w_1)\} \subset W$, such that $w_1 \in V'_x$ and $f^{n_1}(w_1) \in V'_y$. Since $x \in W^u(p)$, there is an integer n_0, such that $f^{-i}(x) \in V$ for any $i \geq n_0$. By the fact that $\mathrm{Orb}^+(f^{N+1}(y)) \cup \mathrm{Orb}^-(f^{-n_0}(x)) \subset V$, and $\overline{V} \cap (\bigcup_{i=0}^N f^i(\overline{V_x})) = \emptyset$, we can see that the piece of orbit segment $\{w_1, f(w_1), \cdots, f^{n_1}(w_1)\}$ is disjoint with $\mathrm{Orb}^+(f^{N+1}(y)) \cup \mathrm{Orb}^-(f^{-n_0}(x))$. Take $L = n_0 + n_1 + N$.

The connecting process: to get a transverse homoclinic point

Now we do the perturbations by Theorem 2.25 and get a homoclinic point of q contained in U_x.

The perturbation at x. The point $f^{-n_0}(x)$ has a positive iterate $x \in V'_x$ and the point $f^{n_1}(w_1)$ has a negative iterate $w_1 \in V'_x$. By Theorem 2.25, there is a diffeomorphism $f_1 \in \mathcal{U}$, such that:

- f_1 coincides with f outside $\bigcup_{i=0}^N f^i(V_x)$,
- $f^{n_1}(w_1)$ is on the positive orbit of $f^{-n_0}(x)$ under f_1,

Then there is a point $z \in V_x$, such that $f^{n_1}(w_1)$ is on the positive orbit of z under f_1 and $f^{-n_0}(x)$ is on the negative orbit of z under f_1. Assume that $f^{n_1}(w_1) = f_1^{m_1}(z)$ and $f^{-n_0}(x) = f_1^{-m_0}(z)$, we have that $m_1 + m_0 \leq n_1 + n_0 + N = L$. Moreover, the negative orbit of $f^{-n_0}(x)$ under f is still the negative orbit of $f^{-n_0}(x)$ under f_1, hence $z \in W^u(q, f_1)$. Also, by the fact that $\bigcup_{i=0}^N f^i(\overline{V_y}) \subset V \setminus \overline{\mathrm{Orb}^-(x)}$, one can see $(\bigcup_{i=0}^N f^i(\overline{V_y})) \cap \overline{\mathrm{Orb}^-(z, f_1)} = \emptyset$.

The perturbation at y. By the above construction, f_1 coincides with f in $\bigcup_{i=0}^N f^i(V_y)$, and, under the iterate of f_1, the point z has a positive iterate $f_1^{m_1}(p) = f^{n_1}(w_1) \in V'_y$ and the point $f^{N+1}(y)$ has a negative iterate $y \in V'_y$. By Theorem 2.25, there is $f_2 \in \mathcal{U}$, such that:

- f_2 coincides with f_1 outside $\bigcup_{i=0}^N f^i(V_y)$, hence f_2 coincides with f outside $(\bigcup_{i=0}^N f^i(V_x)) \cup (\bigcup_{i=0}^N f^i(V_y))$,
- the point $f^{N+1}(y)$ is on the positive iterate of z under f_2.

By the fact that $(\bigcup_{i=0}^N f^i(\overline{V_y})) \cap \overline{\mathrm{Orb}^+(f^{N+1}(y), f)} = \emptyset$, we can see that the positive orbit of $f^{N+1}(y)$ under f is still the positive orbit of $f^{N+1}(y)$ under f_2. Hence $z \in W^s(q, f_2)$. Since $(\bigcup_{i=0}^N f^i(\overline{V_y})) \cap \overline{\mathrm{Orb}^-(z, f_1)} = \emptyset$, we have that $\mathrm{Orb}^-(z, f_1) = \mathrm{Orb}^-(z, f_2)$, hence $z \in W^u(q, f_2)$. Then the point $z \in W^s(q, f_2) \cap W^u(q, f_2)$ is a homoclinic point of the hyperbolic fixed point q of f_2.

By perturbing f_2 to a diffeomorphism $g \in \mathcal{U}$ with an arbitrarily C^1 small perturbation, we can assume that z is a transverse homoclinic point of the hyperbolic fixed point q of g. Moreover, the orbit of z under g is the same as that of f_2, hence the number of points of $\mathrm{Orb}(z, g) \setminus V$ is no more than $m_1 + m_0$, hence no more than L.

Periodic orbits shadowing the orbit of a homoclinic point: end of the proof

Now we have obtained a diffeomorphism $g \in \mathcal{U}$, a hyperbolic fixed point q and a transverse homoclinic point $z \in W^s(q, g) \pitchfork W^u(q, g)$ whose orbit under g has at most L points outside V. Then the set $\mathrm{Orb}(z, g) \cup \{q\}$ is a hyperbolic set. By a standard argument with the λ-lemma and the Smale's homoclinic theorem, for any integer $m \in \mathbb{N}$, there is a periodic point $p \in V_x$ of g, such that $\mathrm{Orb}(p, g)$ has at most L points outside V and has at least m points inside V. The proof of Proposition 3.3 in the periodic case is finished by the fact that $V \subset U$.

The proof of Proposition 3.3 is completed.

3.4 Asymptotic approximation for true orbits: proof of Proposition 3.5

In this section, we give the proof of Proposition 3.5. In fact, the proof is almost the same as the proof of Proposition 10 in [32]. Recall that K is an invariant compact set and $\alpha(x) \subset K$. We consider the hyperbolic periodic point $p \notin K$ with $\overline{W^u(p)} \cap K \neq \emptyset$. We assume that for any point $y \in W^u(p, f)$, we have $\omega(y) \setminus K \neq \emptyset$, otherwise there is nothing needed to prove. Also we assume that $x \notin K$ because the other case can be obtained directly if the proposition is true under the assumption $x \notin K$. We take two steps to get our purpose:

- we choose a sequence of non-periodic points $(z_n)_{n \geq 0}$, such that:

$$z_0 \prec z_1 \prec \cdots \prec K,\ z_0 \in \overline{W^u(p)} \text{ and } z_n \notin \overline{\mathrm{Orb}^-(x)}, \text{ for any } n \geq 0,$$

- then we perturb at every z_n to connect all the points together and avoid $\mathrm{Orb}^-(x)$.

In order to prove Proposition 3.5, we take a decreasing sequence of C^1-neighborhoods $(\mathcal{U}_n)$ of f that satisfies the following properties:

- $\overline{\mathcal{U}_0} \subset \mathcal{U}$,
- the element of $\mathcal{U}_n$ is of the form $f \circ \phi$ with $\phi \in \mathcal{V}_n$, where $(\mathcal{V}_n)$ is a decreasing sequence of C^1 neighborhoods of Id that satisfy the property (F) in Definition 2.24, and $\cap_n \mathcal{V}_n = \{Id\}$.

Then we have that $\cap_n \mathcal{U}_n = \{f\}$. Theorem 2.25 associates to each pair $(f, \mathcal{U}_k)$ a number N_k.

Recall that $\alpha(x) \subset K$ and we have assumed that $x \notin K$. We need the following three lemmas for the proof of Proposition 3.5.

Lemma 3.12. *For any neighborhood W of K, there is a point $z \in \overline{(W \cap W^u(p))} \setminus K$, such that, $z \prec_W K$ and $\mathrm{Orb}^+(z) \subset W$. Moreover, $z \notin \overline{\mathrm{Orb}^-(x)}$.*

Proof. The proof is similar to the proof of Lemma 3.8. We assume W is a small neighborhood of K such that $x \notin \overline{W}$ and $p \notin W$. Take an open neighborhood $V \subset W$ of K, such that $\overline{V} \subset W$. Since $\overline{W^u(p)} \cap K \neq \emptyset$, for any $k \geq 1$, there is a point $x_k \in W^u(p)$ and a positive integer $n(k)$, such that $f^{n(k)}(x_k) \in B(K, \frac{1}{k})$. For k large, the set $B(K, \frac{1}{k})$ is contained in V. We consider the smallest integer $m(k)$ such that the piece of orbit $(f^{m(k)}(x_k), \cdots, f^{n(k)}(x_k))$ is contained in V. By the assumption that $\omega(x_k) \setminus K \neq \emptyset$ for any $k \geq 1$, we can see that both $m(k)$ and $n(k) - m(k)$ go to infinity as k goes to infinity.

By taking convergent subsequence if necessary, assume that the sequence $\{f^{m(k)}(x_k)\}$ converges to a point $z \in \overline{V}$, and $\{f^{n(k)}(x_k)\}$ converges to a point $z' \in K$. It can be obtained directly that $z \in \overline{W^u(p)}$. By a similar argument as in the proof of Lemma 3.8, we can obtain the fact that $z \notin K$, and $z \prec_W z'$. Then we have $z \prec_W K$ since $z' \in K$. Since $f^{m(k)}(x_k), \cdots, f^{n(k)}(x_k)$ is contained in V and $n(k) - m(k)$ go to infinity, we have that $\mathrm{Orb}^+(z) \subset W$. Then by the assumption $x \notin \overline{W}$, we have $z \notin \mathrm{Orb}^-(x)$. Hence $z \notin \overline{\mathrm{Orb}^-(x)}$ because $\overline{\mathrm{Orb}^-(x)} \subset \mathrm{Orb}^-(x) \cup K$. $\square$

Lemma 3.13. *There are a point $y \in W^u(p)$, a sequence of points $(z_k)_{k \geq 1}$, three sequences of neighborhoods $(U_k)_{k \geq 1}$, $(V_k)_{k \geq 1}$, $(W_k)_{k \geq 0}$ and a sequence of finite segment of orbits $Y_k = (y_k, f(y_k), \cdots, f^{m(k)}(y_k))_{k \geq 0}$, such that:*

1. *$W_{k+1} \subset W_k$ and $\cap_k W_k = K$;*
2. *Theorem 2.25 can be applied to $z_k \in V_k \subset U_k$ for the triple $(f, \mathcal{U}_k, N_k)$, and $\overline{f^n(U_k)} \subset W_k \setminus W_{k+1}$ for all $0 \leq n \leq N_k$;*
3. *$\overline{U_k} \cap \mathrm{Orb}^-(x) = \emptyset$, for any $k \geq 1$;*
4. *$z_k \prec_{W_k} z_{k+1}$ and $z_k \prec_{W_k} K$, for any $k \geq 1$;*
5. *the points $f^{m(k)}(y_k)$ and y_{k+1} are contained in V_{k+1} for all $k \geq 0$ where $y_0 = y$, and $\mathrm{Orb}^-(y) \cap W_1 = \emptyset$;*
6. *$Y_k \subset W_k \setminus W_{k+2}$ for all $k \geq 0$ and $Y_k \cap \mathrm{Orb}^-(x) = \emptyset$, for all $k \geq 0$;*

Proof. We build all the sequences by induction. Set $W_0 = M$. We first choose W_1, z_1, V_1, U_1 and Y_0.

Since all periodic orbits in K are hyperbolic, there is a neighborhood W_1 of K, such that there is no periodic points with period less than or equal to N_1 contained in $W_1 \setminus K$. Also we can assume that $p \notin \overline{W_1}$. By Lemma 3.12, there is $z_1 \in W_1 \setminus K$, such that $z_1 \prec_{W_1} K$, $z_1 \notin \overline{\mathrm{Orb}^-(x)}$, and $z_1 \in \overline{W^u(p)}$. By the choice of W_1 and by the fact $z_1 \in W_1 \setminus K$, there is a neighborhood U_1 of z_1, that is disjoint from its N_1 first iterates and $\overline{f^n(U_1)} \subset W_1 \setminus K$ for any $0 \leq n \leq N_1$. Moreover, because $z_1 \notin \overline{\mathrm{Orb}^-(x)}$, we can assume $\overline{U_1} \cap (\mathrm{Orb}^-(x) \cup K) = \emptyset$. By Theorem 2.25, there is $V_1 \subset U_1$ associated to $(f, \mathcal{U}_1, N_1)$. Then there are a point $y \in W^u(p) \setminus W_1$ and a positive integer $m(0)$, such that $f^{m(0)}(y) \in V_1$. Moreover, by considering a negative iterate of y if necessary, we can assume that $\mathrm{Orb}^-(y) \cap W_1 = \emptyset$. We take $y_0 = y$ and $Y_0 = (y, f(y), \cdots, f^{m(0)}(y))$. To sum up, we have obtained W_1, z_1, V_1, U_1 and Y_0.

Now we construct the sequences by induction on k. After W_k, z_k, V_k, U_k and Y_{k-1} have been built, there is $W_{k+1} \subset W_k$ such that

- there is no periodic point with period less than or equal to N_{k+1} contained in $W_{k+1} \setminus K$;
- $\overline{f^n(U_k)} \cap W_{k+1} = \emptyset$, for all $1 \leq n \leq N_k$;
- $W_{k+1} \cap Y_{k-1} = \emptyset$;
- W_{k+1} is contained in $B(K, \frac{1}{k})$.

By Lemma 2.14, there is $z_{k+1} \in W_{k+1} \setminus K$, such that $z_k \prec_{W_k} z_{k+1} \prec_{W_{k+1}} K$ and $\mathrm{Orb}^+(z_{k+1}) \subset W_{k+1}$. Since $\mathrm{Orb}^+(z_{k+1}) \subset W_{k+1}$ and $x \notin W_{k+1}$, we have that $z_{k+1} \notin \mathrm{Orb}^-(x)$. Moreover, by the fact that $z_{k+1} \notin K$ and $\alpha(x) \subset K$, we have that $z_{k+1} \notin \overline{\mathrm{Orb}^-(x)}$. By Theorem 2.25, there are neighborhoods $V_{k+1} \subset U_{k+1}$ of z_{k+1} associated to $(f, \mathcal{U}_{k+1}, N_{k+1})$, such that:

- $\overline{U_{k+1}} \cap (\mathrm{Orb}^-(x) \cup K) = \emptyset$,
- $\overline{f^n(U_{k+1})} \subset W_{k+1}$ for all $0 \leq n \leq N_{k+1}$.

Then there is $Y_k = (y_k, f(y_k), \cdots, f^{m(k)}(y_k))$, such that $y_k \in V_k$ and $f^{m(k)}(y_k) \in V_{k+1}$. Since $\overline{U_k} \cap \mathrm{Orb}^-(x) = \emptyset$, we have that $y_k \notin \mathrm{Orb}^-(x)$, and hence Y_k is disjoint from $\mathrm{Orb}^-(x)$.

Then we finish the proof of Lemma 3.13. □

Remark 3.14. *We point out here that, in Lemma 3.13, there is an open set V containing $\mathrm{Orb}^-(x)$, such that $V \cap U_k = \emptyset$ for all $k \geq 1$. In fact, since $x \notin K$, for any integer $n \in \mathbb{N}$, there is n_k, such that $f^{-n}(x) \notin W_{n_k}$. By the fact that $U_k \subset W_k$ and $\overline{U_k} \cap \mathrm{Orb}^-(x) = \emptyset$, for any $n \geq 0$, there is an open neighborhood B_n of $f^{-n}(x)$, such that $B_n \cap U_k = \emptyset$ for any $k \geq 1$. Then we take $V = \bigcup_{n \geq 0} B_n$.*

Now we fix the point $y \in W^u(p)$, the open set V and the sequences $(z_k)_{k\geq 1}$, $(U_k)_{k\geq 1}$, $(V_k)_{k\geq 1}$, $(W_k)_{k\geq 0}$ and $(Y_k)_{k\geq 0}$ as in Lemma 3.13. We have the following lemma.

Lemma 3.15. *There are a sequence of perturbations $(g_k)_{k\geq 0}$ of f and a strictly increasing sequence of integers $(n_k)_{k\geq 0}$, such that,*

1. *$g_0 = f$ and $n_0 = 0$;*
2. *there is $\phi_k \in \mathcal{V}_k$, such that $\phi_k|_{M\setminus(U_k\cup\cdots\cup f^{N_k-1}(U_k))} = Id|_{M\setminus(U_k\cup\cdots\cup f^{N_k-1}(U_k))}$ and $g_k = g_{k-1}\circ\phi_k$, for $k \geq 1$;*
3. *for any $l = \{0, 1, \cdots, k-1\}$, the piece of orbit $(g_k^{n_l}(y), g_k^{n_l+1}(y), \cdots, g_k^{n_{l+1}}(y))$ is contained in $W_l \setminus W_{l+2}$.*

Proof. We build inductively the sequences (g_k) and (n_k) and another sequence of integers $(m_k)_{k\geq 0}$ which satisfy the conclusions and also the following properties:

- $m_k > n_k$ and $g_k^{m_k}(y) \in V_{k+1}$;
- the piece of orbit $(g_k^{n_k}(y), g_k^{n_k+1}(y), \cdots, g_k^{m_k}(y))$ is contained in $W_k\setminus W_{k+2}$.

First, we take $g_0 = f$ and $n_0 = 0$. By Lemma 3.13, there is $m_0 > 0$, such that $g_0^{m_0}(y) \in V_1$ and the piece of orbit $(y = g_0^{n_0}(y), g_0(y), \cdots, g_0^{m_0}(y))$ is contained in $W_0 \setminus W_2$.

Now assume that g_k, n_k and m_k have been built, we explain how to get g_{k+1}, n_{k+1} and m_{k+1}.

The forward orbit of $g_k^{n_k}(y)$ has a positive iterate $g_k^{m_k}(y) \in V_{k+1}$, and the backward orbit of $f^{m(k+1)}(y_{k+1})$ has a negative iterate $y_{k+1} \in V_{k+1}$. Moreover, these segments of orbit are contained in $W_k \setminus W_{k+3}$. Since g_k coincides with f on the set $U_{k+1} \cup f(U_{k+1}) \cup \cdots \cup f^{N_{k+1}}(U_{k+1})$, one can apply Theorem 2.25 to $(g_k, \mathcal{U}_{k+1}, V_{k+1}, U_{k+1})$ and get a diffeomorphism g_{k+1}. The new diffeomorphism g_{k+1} is of the form $g_k \circ \phi_{k+1}$, where $\phi_{k+1}|_{M\setminus(U_{k+1}\cup\cdots\cup f^{N_{k+1}-1}(U_{k+1}))} = Id|_{M\setminus(U_{k+1}\cup\cdots\cup f^{N_{k+1}-1}(U_{k+1}))}$ and $f\circ\phi_{k+1} \in \mathcal{U}_{k+1}$, thus $\phi_{k+1} \in \mathcal{V}_{k+1}$.

Since $g_{k+1}|_{M\setminus(U_{k+1}\cup\cdots\cup f^{N_{k+1}-1}(U_{k+1}))} = g_k|_{M\setminus(U_{k+1}\cup\cdots\cup f^{N_{k+1}-1}(U_{k+1}))}$, the piece of orbit $(y, g_k(y), \cdots, g_k^{n_k}(y))$ under g_k coincides with the one $(y, g_{k+1}(y), \cdots, g_{k+1}^{n_k}(y))$. By the new diffeomorphism g_{k+1}, the forward orbit of $g_{k+1}^{n_k}(y)$ has an iterate $f^{m(k+1)}(y_{k+1})$ under g_{k+1} contained in V_{k+2}. That is to say, there is an integer $m_{k+1} > n_k$, such that $f^{m(k+1)}(y_{k+1}) = g_{k+1}^{m_{k+1}}(y)$. Moreover, there exists an integer n_{k+1} with $n_k < n_{k+1} < m_{k+1}$, such that:

– the piece of orbit $(g_{k+1}^{n_k}(y), \cdots, g_{k+1}^{n_{k+1}}(y))$ is contained in the union of $\{g_k^{n_k}(y), \cdots, g_k^{m_k}(y)\}$ and $U_{k+1}\cup\cdots\cup f^{N_{k+1}-1}(U_{k+1})$, hence it is contained in $W_k \setminus W_{k+2}$,

– the piece of orbit $(g_{k+1}^{n_{k+1}}(y), \cdots, g_{k+1}^{m_{k+1}}(y))$ is contained in the union of $U_{k+1} \cup \cdots \cup f^{N_{k+1}-1}(U_{k+1})$ and $\{y_{k+1}, \cdots, f^{m(k+1)}(y_{k+1})\}$, hence it is contained in $W_{k+1} \setminus W_{k+3}$.

Then the conclusions are satisfied for $k+1$. This ends the proof of Lemma 3.15. □

End of the proof of Proposition 3.5. Since the supports $U_i \cup \cdots \cup f^{N_i-1}(U_i)$ and $U_j \cup \cdots \cup f^{N_j-1}(U_j)$ of the perturbations ϕ_i and ϕ_j are disjoint for any $i \neq j$, and $(\mathcal{V}_n)$ satisfy the property (F), then the sequence $g_k = f \circ \phi_1 \circ \cdots \circ \phi_k$ converges to a diffeomorphism $g \in \mathcal{U}_0 \subset \mathcal{U}$. By the constructions, g coincides with $f \circ \phi_k$ in the set $U_k \cup \cdots \cup f^{N_k-1}(U_k)$ and with f elsewhere. We take V to be the neighborhood of $\mathrm{Orb}^-(x)$ in Remark 3.14, then it holds that g coincides with f on the set $\mathrm{Orb}(p) \cup K \cup V \cup \mathrm{Orb}^-(y)$ and $\omega(y, g) \subset K$. Since $\mathrm{Orb}^-(x) \subset V$, we have that Dg coincides with Df on $\mathrm{Orb}^-(x)$. Moreover, since g is the limit of the sequence (g_k), by Lemma 3.15, for any $n > n_k$, $g^n(y) \in W_k$. Then we have that $\omega(y, g) \subset K$. This finishes the proof of Proposition 3.5. □

3.5 Asymptotic approximation for pseudo-orbits: proof of Proposition 3.6

To prove Proposition 3.6, we use the techniques of [15, 32] to get true orbits by perturbing a pseudo-orbit. Similarly to the proof of Proposition 3.5, we have to perturb infinitely many times in a special neighborhood to keep some part of the initial dynamic unchanged. The proof refers a lot to [15] and Section 3.2 of [35].

We take several steps. First, we choose an open set that covers all positive orbits of X that are not on the local stable manifold of periodic orbits with small periods. Actually, we choose a special topological tower for X. Second, we construct a sequence of disjoint perturbation domains containing in their interior the special topological tower. Then, we choose an infinitely long pseudo-orbit in X that goes from z to K, has jumps only in the perturbation domains and accumulates to K in the future. Finally, we perturb in the perturbation domains to construct a true orbit which goes from z to K and accumulates to K in the future.

We take a C^1 neighborhood $\mathcal{U}_0$ of the diffeomorphism f_0 with $\overline{\mathcal{U}_0} \subset \mathcal{U}$, such that, the element of $\mathcal{U}_0$ is of the form $f \circ \phi$ with $\phi \in \mathcal{V}_0$, where $\mathcal{V}_0$ is a C^1-neighborhood of Id that satisfies the property (F) in Definition 2.24. Then there is a smaller C^1 neighborhood $\mathcal{U}' \subset \mathcal{U}_0$ of f_0 and an integer N_0 associated to $(f_0, \mathcal{U}_0)$ by Theorem 2.26. Take the integer $T = 10\kappa_d d N_0$ where the integer $\kappa_d \geq 1$ is the number given by Lemma 2.36.

From now on, we fix the C^1 neighborhoods $\mathcal{U}' \subset \mathcal{U}_0$ of f_0 and the integer T. Consider a diffeomorphism $f \in \mathcal{U}'$, an invariant compact set K, a positive invariant compact set X and a point z, satisfying the following properties:

- all periodic points contained in K are hyperbolic,
- all periodic points contained in X with period less than or equal to T are hyperbolic,
- for any $\varepsilon > 0$, there is an ε-pseudo-orbit contained in X that connects from z to K.

Also we fix a neighborhood U of $X \setminus K$.

We take a decreasing sequence of C^1-neighborhoods $(\mathcal{U}_n)_{n\geq 1}$ of f such that, $\mathcal{U}_1 \subset \mathcal{U}_0$, and $\cap_n \mathcal{U}_n = \{f\}$. Moreover, the element of $\mathcal{U}_n$ is of the form $f \circ \phi$ with $\phi \in \mathcal{V}_n$, where $(\mathcal{V}_n)$ is a decreasing sequence of C^1 neighborhoods of Id that satisfy the property (F) in Definition 2.24. For any $k \geq 1$, Theorem 2.25 associates to each pair $(f,\mathcal{U}_k)$ an integer N_k. We can assume that $(N_k)_{k\geq 0}$ is an increasing sequence. We assume also that $z \notin K$, otherwise, there is noting to prove. For an integer N, denote by $Per_N(f)$ the set of periodic points of f whose period is no more than N. Fix a small number $\gamma > 0$ (to determine the C^0 distance between the new created diffeomorphism and f).

3.5.1 Choice of topological towers

In this section, we construct a family of special topological towers for the set X with the properties stated in Lemma 3.16.

Lemma 3.16. *For any $\delta > 0$, for any decreasing sequence of positive constants $(\gamma_n)_{n\geq 0}$, and for any increasing sequence of integers $(L_k)_{k\geq 0}$ where $L_0 = 10dN_0$, there are a decreasing sequence of neighborhoods $(U_k)_{k\geq 0}$ of K, a sequence of open sets $(W_k)_{k\geq 0}$, and a sequence of compact sets $(D_k)_{k\geq 0}$, such that, putting $X_k = X \cap \overline{(U_k \setminus U_{k+1})}$ for all $k \geq 0$, the following properties are satisfied.*

- *$U_0 = M$, $z \notin \overline{U_1}$, and $\bigcap_{k\geq 0} U_k = K$,*
- *For any $k \geq 0$,*
  1. *there is no periodic orbit with period less than $\kappa_d^2 L_{k+1}$ contained in $\overline{U_{k+1}} \setminus K$,*
  2. *$f^i(\overline{U_{k+1}}) \subset U_k$, for all $-4\kappa_d^2 L_{k+1} \leq i \leq 4\kappa_d^2 L_{k+1}$,*
  3. *the L_k sets $\overline{W_k}, f(\overline{W_k}), \cdots, f^{L_k-1}(\overline{W_k})$ are pairwise disjoint, contained in $U \setminus K$, and also contained in $U_{k-1} \setminus \overline{U_{k+2}}$, where we put $U_{-1} = U_0 = M$,*
  4. *$f^i(\overline{U_{k+2}}) \cap \overline{(W_0 \cup \cdots \cup W_k)} = \emptyset$, for any $-4\kappa_d^2 L_{k+2} \leq i \leq 4\kappa_d^2 L_{k+2}$,*

5. *for any $l < k$, any $0 \le i \le L_l$ and any $0 \le j \le L_k$, we have that $\overline{f^i(W_l)} \cap \overline{f^j(W_k)} = \emptyset$,*
6. *the set D_k is contained in W_k, satisfying that, any point in $X_0 \setminus (\bigcup_{p \in Per_{L_0}(f)} W^s_\delta(p))$ has a positive iterate in $\mathrm{Int}(D_0)$, and for $k \ge 1$, any point in X_k has a positive iterate in $\mathrm{Int}(D_k \cup D_{k-1})$,*
7. *the diameter of every connected component of W_k is smaller than γ_k.*

Remark 3.17. *The set $W_0 \cup \cdots \cup W_k$ can be seen as a special topological tower for $X_0 \cup \cdots \cup X_k$, from the items 3, 5, and 6.*

Proof. We build inductively the sequences $(U_k)_{k\ge0}$, $(W_k)_{k\ge0}$ and $(D_k)_{k\ge0}$ from a sequence of open sets $(W'_k)_{k\ge0}$ and a sequence of compact sets $(D'_k)_{k\ge0}$, which satisfy the following additional properties: for any $k \ge 0$,

- $1'$. *the set W_k is contained in a small neighborhood of W'_k, and $W'_k \subset W_k$,*
- $2'$. *the sets $\overline{W'_k}, f(\overline{W'_k}), \cdots, f^{L_k-1}(\overline{W'_k})$ are pairwise disjoint, contained in $U \setminus K$, and also contained in $U_{k-1} \setminus \overline{U_{k+2}}$, (notice that $U_{-1} = M$),*
- $3'$. *for all $-4\kappa_d^2 L_{k+2} \le i \le 4\kappa_d^2 L_{k+2}$, we have $f^i(\overline{U_{k+2}}) \cap \overline{(W'_0 \cup \cdots \cup W'_k)} = \emptyset$,*
- $4'$. *for any $l < k$, any $0 \le i \le L_l$ and any $0 \le j \le L_k$, we have $\overline{f^i(W_l)} \cap \overline{f^j(W'_k)} = \emptyset$,*
- $5'$. $D_k = (D'_k \cup D'_{k+1}) \cap W_k$, where $D'_0 \subset W'_0$ and $D'_{k+1} \subset W_k \cup W'_{k+1}$,
- $6'$. *any point in $X_0 \setminus (\bigcup_{p \in Per_{L_0}(f)} W^s_\delta(p))$ has a positive iterate in $\mathrm{Int}(D'_0)$ and any point in X_k has a positive iterate contained in $\mathrm{Int}(D'_k)$ for any $k \ge 1$,*
- $7'$.*the diameter of every connected component of W'_k is smaller than $\frac{\gamma_k}{2}$.*

Put $U_{-1} = U_0 = M$. We construct inductively the sets U_{k+1}, W'_k, D'_k, W_{k-1} and D_{k-1}.

The sets U_1, W'_0, and D'_0: the case where $k = 0$. By the assumption of hyperbolicity of periodic orbits in K, we can take a neighborhood $U_1 \subset B(K, 1)$ of K such that $z \notin \overline{U_1}$ and there is no periodic orbit with periodic less than $\kappa_d^2 L_1$ in $\overline{U_1} \setminus K$. Notice that $U_0 = M$. The properties 1 and 2 are satisfied.

Recall that X contains no non-hyperbolic periodic orbit with period less than T, where $T = 10\kappa_d d N_0 = \kappa_d L_0$. Hence X_0 contains no non-hyperbolic periodic orbit with period less than $\kappa_d L_0$. By Lemma 2.36, there are an open set $W'_0 \subset U$ whose closure $\overline{W'_0}$ is a compact d-dimensional sub-manifold with boundary and a compact set $D'_0 \subset W'_0$, such that $\overline{W'_0}$ is disjoint from its L_0 iterates, which is the first property of item $2'$. Moreover, any point in $X_0 \setminus (\bigcup_{p\in Per_{L_0}(f)} W^s_\delta(p))$ has a positive iterate contained in $\mathrm{Int}(D'_0)$ and hence the item $6'$ is satisfied. By the item 4 of Lemma 2.36, the set $\overline{W'_0}$ is contained in a small neighborhood of $X_0 \cup f(X_0) \cup \cdots \cup f^{\kappa_d L_0}(X_0)$. Moreover, we can choose W'_0 such that the diameter of every connected component of W'_0 is smaller than $\frac{\gamma_0}{2}$, which is the item $7'$. Hence we can assume that $\bigcup_{i=0}^{L_0} f^i(\overline{W'_0}) \subset U \setminus K$, since X is positively invariant. To make the sequences complete, we could put $D'_{-1} = D_{-1} = W_{-1} = \emptyset$. Notice that, we do not need to check other items for the case $k = 0$.

The sets U_{k+2}, W'_{k+1}, D'_{k+1}, W_k and D_k. Assume U_{j+1}, W'_j, D'_j, W_{j-1} and D_{j-1} have been constructed for any $0 \le j \le k$. Now we build the sets U_{k+2}, W'_{k+1}, D'_{k+1}, W_k and D_k.

We take a neighborhood $U_{k+2} \subset U_{k+1} \cap B(K, \frac{1}{k+2})$ of K, such that:

- there is no periodic orbit with periodic less than $\kappa_d^2 L_{k+2}$ in $\overline{U_{k+2}} \setminus K$, which is the property 1,
- $f^i(\overline{U_{k+2}}) \subset U_{k+1}$, for all $-4\kappa_d^2 L_{k+2} \le i \le 4\kappa_d^2 L_{k+2}$, which is the property 2,
- $\overline{W'_k} \cap f^i(\overline{U_{k+2}}) = \emptyset$ for all $-4\kappa_d^2 L_{k+2} \le i \le 4\kappa_d^2 L_{k+2}$, which implies the item $3'$ and the last property of item $2'$.

Consider the compact set $X_{k+1} = X \cap \overline{U_{k+1} \setminus U_{k+2}}$. Notice that X_{k+1} contains no periodic orbit of period less than or equal to $\kappa_d^2 L_{k+1}$. By Lemma 2.36, there is an open set V'_{k+1} whose closure $\overline{V'_{k+1}}$ is a compact d-dimensional submanifold with boundary such that any point in X_{k+1} has a positive iterate contained in V'_{k+1}. Moreover, the set $\overline{V'_{k+1}}$ is disjoint from its first $\kappa_d L_{k+1}$ first iterates and $\overline{V'_{k+1}}$ is contained in a small neighborhood of $\bigcup_{i=0}^{\kappa_d^2 L_{k+1}} f^i(X_{k+1})$. By taking this neighborhood small, we can assume that V'_{k+1} satisfies the following properties:

- Since $f^i(\overline{U_{k+1}}) \subset U_k$ for any $-4\kappa_d^2 L_{k+1} \le i \le 4\kappa_d^2 L_{k+1}$, and $X_{k+1} \subset \overline{U_{k+1}}$, we have that $f^i(\overline{V'_{k+1}}) \subset U_k$, for any $-2\kappa_d^2 L_{k+1} \le i \le 2\kappa_d^2 L_{k+1}$.
- Since $f^i(\overline{U_{k+1}}) \cap (\overline{W_0 \cup \cdots \cup W_{k-1}}) = \emptyset$ for any $-4\kappa_d^2 L_{k+1} \le i \le 4\kappa_d^2 L_{k+1}$, we have that $f^i(\overline{V'_{k+1}}) \cap (\overline{W_0 \cup \cdots \cup W_{k-1}} \cup K) = \emptyset$ for any $-2\kappa_d^2 L_{k+1} \le i \le 2\kappa_d^2 L_{k+1}$.

Moreover, we can choose V'_{k+1} such that the diameter of all its connected components is small enough, such that all the i^{th} iterates of every connected component of V'_{k+1} is of diameter smaller than $\frac{\gamma_{k+1}}{3}$, for any $0 \le i \le \kappa_d L_{k+1}$.

Recall that $\overline{W'_k}$ and $\overline{V'_{k+1}}$ are two compact d-dimensional sub-manifolds with boundary and $\overline{V'_{k+1}}$ is disjoint from its first $\kappa_d L_{k+1}$ first iterates. By Lemma 2.34, considering $\overline{W'_k}$ and $\overline{V'_{k+1}}$ as W' and V', and considering the integer L_{k+1} as the integer T, there is an open set $S_k = W_k \cup V_{k+1}$ satisfying the following properties:

- $\overline{W_k}$ and $\overline{V_{k+1}}$ are two compact d-dimensional sub-manifolds with boundary.
- $W_k \cup V_{k+1} \subset U \setminus K$.
- W_k is a small neighborhood of W'_k, and hence $\overline{W_k}$ is disjoint with its first L_k iterates. Moreover, the properties $2'$, $3'$, $4'$ of W'_k implies the properties of 3, 4, 5 of W_k. The property $1'$ is automatically satisfied.
- $V'_{k+1} \subset \bigcup_{i=0}^{\kappa_d L_{k+1}} f^{-i}(S_k)$,
- $\overline{W_k} \cap f^i(\overline{V_{k+1}}) = \emptyset$ for all $i = 0, \pm 1, \cdots, \pm L_{k+1}$,
- $\overline{V_{k+1}}$ is contained in a small neighborhood of $V'_{k+1} \cup f(V'_{k+1}) \cup \cdots \cup f^{\kappa_d L_{k+1}}(V'_{k+1})$ and disjoint from its L_{k+1} iterates. Thus we can assume that $K \cap \overline{V_{k+1}} = \emptyset$, and for all $-\kappa_d L_{k+1} \leq i \leq \kappa_d L_{k+1}$, we have $f^i(\overline{V_{k+1}})) \cap \overline{(W_0 \cup \cdots \cup W_{k-1})} = \emptyset$ and $f^i(\overline{V_{k+1}})) \subset U_k$.

Moreover, by the assumption of the diameter of every connected component of W'_k and V'_{k+1}, we can take W_k and V_{k+1} such that every connected component of W_k is of diameter less than γ_k and every connected component of V_{k+1} is of diameter less than $\frac{\gamma_k}{2}$. Then the item 7 is satisfied.

By the fact that any point in X_{k+1} has a positive iterate contained in V'_{k+1}, and $V'_{k+1} \subset \bigcup_{i=0}^{\kappa_d L_{k+1}} f^{-i}(V_{k+1})$, one can see that any point in X_{k+1} has a positive iterate contained in S_k. By the compactness of X_{k+1}, there is a compact set $D'_{k+1} \subset S_k$, such that all such iterates are contained in $\mathrm{Int}(D'_{k+1})$. Put $W'_{k+1} = V_{k+1}$ and $D_k = (D'_k \cup D'_{k+1}) \cap W_k$. Then we have $D'_{k+1} \subset S_k = W_k \cup W'_{k+1}$. From the construction of V_{k+1}, we can see that W'_{k+1} and D'_{k+1} satisfy the properties $6, 2', 3', 4', 5', 6', 7'$.

This finishes the construction of the sets U_{k+2}, W'_{k+1}, D'_{k+1} W_k and D_k.

Notice that $\bigcap_{k\geq 0} U_k = K$ and $z \notin \overline{U_1}$ are obviously satisfied by the choice of U_k. This finishes the proof of Lemma 3.16. □

3.5.2 Construction of perturbation domains

We take $L_k = 10dN_k$ for all $k \geq 0$, and take a small number $\delta > 0$, such that for any two different hyperbolic periodic points $q_1, q_2 \in Per_{N_0}(f) \cap X$, we have $W^{\sigma_1}_\delta(q_1) \cap W^{\sigma_2}_\delta(q_2) = \emptyset$, where $\sigma_i \in \{u, s\}$. By Lemma 3.16, we get the sequences $(U_k)_{k\geq 0}$, $(W_k)_{k\geq 0}$ and $(D_k)_{k\geq 0}$. We still denote $X_k = X\cap\overline{(U_k \setminus U_{k+1})}$ for all $k \geq 0$.

Now we build the perturbation domains for the family (X_k). The techniques are mainly from Section 4.1 and 4.2 of [15]. First, we build the perturbation domains that covers the points which are not on the local stable manifolds of periodic orbits with period less than or equal to N_0. The proof is essentially due to Corollaire 4.1 of [15]. They deal with a family of perturbation domains with the same order, thus the union forms a perturbation domain. Here we have a sequence of perturbation domains with different orders, however, the construction of each perturbation domain can be separated.

Lemma 3.18. *There is a perturbation domain B_k of order N_k for $(f,\mathcal{U}_k)$ for each $k \geq 0$, such that the sequence $(B_k)_{k\geq 0}$ satisfies the following properties.*

1. *The supports of the perturbations domains B_k are pairwise disjoint, contained in U, and also contained in $U_{k-1} \setminus \overline{U_{k+2}}$.*
2. *Any point of $X_0\setminus(\bigcup_{p\in Per_{N_0}(f)} W^s_\delta(p))$ has a positive iterate in the interior of one tile of the perturbation domain B_0 and any point of X_k has a positive iterate in the interior of one tile of the perturbation domain $B_{k-1}\cup B_k$ for $k \geq 1$.*

In consequence, for any $k \geq 0$, there is a finite family of tiles $\mathcal{C}_k$ associated to B_k, and a family of compact sets $\mathcal{D}_k$ contained in the interior of tiles of $\mathcal{C}_k$, such that:

- *each tile of $\mathcal{C}_k$ contains exactly one element of $\mathcal{D}_k$, for all $k \geq 0$ and each element of $\mathcal{D}_k$ is contained in a tile of $\mathcal{C}_k$,*
- *any point of $X_0\setminus(\bigcup_{p\in Per_{N_0}(f)} W^s_\delta(p))$ has a positive iterate in the interior of one element of $\mathcal{D}_0$ and any point of X_k has a positive iterate in the interior of one element of $\mathcal{D}_{k-1}\cup\mathcal{D}_k$ for $k \geq 1$.*

Moreover, the diameter of any connected component of B_k is smaller than γ.

Proof. Consider the sequence of open sets $(W_k)_{k\geq 0}$ and the sequence of compact sets $(D_k)_{k\geq 0}$ obtained by Lemma 3.16. Moreover, by the item 7 of Lemma 3.16, the diameters of components of each W_k can be chosen small enough such that all their first L_k iterates are contained in a perturbation domain of order L_k by Theorem 2.32.

Assume W is a component of W_k, and put $D = D_k\cap W$. By assumption, W is contained in a chart of perturbation $\varphi : W \to \mathbb{R}^d$. We can tile W with tiles of proper size such that any cube that intersects $\varphi(D)$ is contained in $\varphi(W)$. We do the same thing for all other components of W_k that has non-empty intersection with D_k and we get a finite family $\mathcal{P}_0$ of perturbation domains, each of them being an open set, pairwise disjoint, contained in W_k, and the union of their closure contains D_k in its interior. Denote Φ_0 the family of perturbation charts in the construction of $\mathcal{P}_0$.

Repeat the construction for $f^{2iN_k}(W_k)$ and $f^{2iN_k}(D_k)$, $i \in \{1, \cdots, 5d-1\}$, and we get the families $\mathcal{P}_i$ of perturbation domains contained in $f^{2iN_k}(W_k)$, pairwise disjoint and the union of their closure contains $f^{2iN_k}(D_k)$ in its interior. Denote Φ_i the family of perturbation charts corresponding to $\mathcal{P}_i$. Consider the family $f^{-2iN_k}(\mathcal{P}_i)$ contained in W_k. The union of the closure of all cubes of $f^{-2iN_k}(\mathcal{P}_i)$ contains D_k in its interior. By a C^1 small perturbation of Φ_i, we can suppose that a point in D_k can only be contained on the boundary of at most d different cubes of all cubes contained in $\cup_{i=0}^{5d-1} f^{-2iN_k}(\mathcal{P}_i)$ [1]. Since there are at least $5d$ families of cubes, we get that any point of D_k is contained in the interior of at least $4d$ families of such cubes.

We replace every cube in $\mathbb{R}^d$ by another one with the same center and homothetic with rate $\rho < 1$ close to 1. Then we get the families $\mathcal{P}_{i,\rho}$ of perturbation domains whose closures are pairwise disjoint. If we choose ρ close enough to 1, then any point of D_k is still contained in the interior of a cube of at least $4d$ families of $(f^{-2dN_k}(\mathcal{P}_{i,\rho}))_{0\leq k\leq 5d-1}$. By the compactness of D_k, for each i, there is a finite family Γ_i of tiles of the domains $f^{-2iN_k}(\mathcal{P}_{i,\rho})$, such that the union Σ_i of the tiles of Γ_i satisfies: any point of D_k is contained in the interior of at least $4d$ compact $(f^{-2iN_k}(\Sigma_i))_{0\leq k\leq 5d-1}$.

By another C^1 small perturbation of Φ_i, we can suppose that any point of D_k is contained on the boundary of the tiles of at most d families of $(f^{-2iN_k}(\Gamma_i))_{0\leq k\leq 5d-1}$. Any point is contained in at least $4d$ families of tiles, hence any point is contained in the interior of at least one of these tiles. Define B_k and $\mathcal{C}_k$ to be the union of the families $\mathcal{P}_{i,\rho}$ and the union of the families Γ_i respectively.

Then the compact set D_k is covered by the interior of the tiles of the family $f^{-2iN_k}(\Gamma_i)$. We can take all the components of the intersection of $f^{2iN_k}(D_k)$ and the elements of the family Γ_i, and this is the family $\mathcal{D}_k$.

Finally, by the assumption that $L_k = 10dN_k$ and the choice of W_k in Lemma 3.16, the supports of perturbation domains $(B_k)_{k\geq 0}$ are pairwise disjoint and are contained in U. Moreover, the support of the perturbation domain B_k is also contained in $U_{k-1}\setminus\overline{U_{k+2}}$. This finishes the proof of Lemma 3.18. □

We also have to construct perturbation domains that cover the stable and unstable manifolds of periodic orbits contained in $X_0 \cap Per_{N_0}(f)$. By the assumption of hyperbolicity of periodic orbits, $X_0 \cap Per_{N_0}(f)$ is a finite set. By Proposition 4.2 in [15], we can construct in the following way.

Lemma 3.19 (Proposition 4.2 of [15])**.** *For any periodic orbit $Q \subset X \cap Per_{N_0}(f)$, any neighborhood V of Q, there are a neighborhood W of Q, two*

[1] In [15], they call the sets of $\cup_{i=0}^{5d-1} f^{-2iN_k}(\mathcal{P}_i)$ on general position. We do not introduce this definition in our paper. The reader can refer to Section 3.3 of [15] for more details.

perturbation domains B_s and B_u of order N_0 for $(f,\mathcal{U}_0)$, two finite families of tiles $\mathcal{C}_s$ and $\mathcal{C}_u$ associated to B_s and B_u respectively, two finite families of compact sets $\mathcal{D}_s$ and $\mathcal{D}_u$, and an integer $n_0(Q)$, such that:

1. *V contains $\overline{W}$ and $\bigcup_{0\leq i\leq N_0-1} f^i(B_s\cup B_u)$.*
2. *$f^i(B_s)\cap f^j(B_u)=\emptyset$ for all $0\leq i,j\leq N_0-1$,*
3. *each element of $\mathcal{D}_s$ is contained in the interior of an element of $\mathcal{C}_s$, and each element of $\mathcal{D}_u$ is contained in the interior of an element of $\mathcal{C}_u$. Moreover, each tile of $\mathcal{C}_s$ and $\mathcal{C}_u$ contains exactly an element of $\mathcal{D}_s\cup\mathcal{D}_u$*
4. *for any two pairs $D_s\in\mathcal{D}_s$ and $D_u\in\mathcal{D}_u$, there is $n\in\{0,\cdots,n_0(Q)\}$, such that $f^n(D_s)\cap D_u\neq\emptyset$.*
5. *for any point $z\in W\setminus W^s_{loc}(Q)$, there is $n>0$ and $D\in\mathcal{D}_u$, such that $f^n(z)\in\operatorname{Int}(D)$ and $f^i(z)\in V$ for all $0\leq i\leq n$. Moreover, if $f(z)\notin W$, then $n\leq n_0(Q)$.*
6. *for any point $z\in W\setminus W^u_{loc}(Q)$, there is $n>0$ and $D\in\mathcal{D}_s$, such that $f^{-n}(z)\in\operatorname{Int}(D)$ and $f^{-i}(z)\in V$ for all $0\leq i\leq n$. Moreover, if $f^{-1}(z)\notin W$, then $n\leq n_0(Q)$.*

3.5.3 Choice of a pseudo-orbit

By Lemma 3.18, we have the sequences of perturbation domains $(B_k)_{k\geq 0}$, tiles $(\mathcal{C}_k)_{k\geq 0}$ and families of compact sets $(\mathcal{D}_k)_{k\geq 0}$. Since there are only finitely many periodic orbits contained in $Per_{N_0}(f)\cap X$, and they are all outside $\overline{U_1}$, we can take for each periodic orbit $Q\subset Per_{N_0}(f)\cap X$ an open neighborhood $V(Q)\subset U$ that are pairwise disjoint, disjoint from $\overline{U_1}$ and disjoint from $f^i(B_k)$ for any $0\leq i\leq N_k-1$ and any $k\geq 0$. By Lemma 3.19, we have for each Q the open set $W(Q)$, the perturbation domains $B_s(Q)$ and $B_u(Q)$, the families of tiles $\mathcal{C}_s(Q)$ and $\mathcal{C}_u(Q)$, the families of compact sets $\mathcal{D}_s(Q)$ and $\mathcal{D}_u(Q)$ and the number $n_0(Q)$. By the choice of $V(Q)$, we have that $f^i(B_\sigma(Q))\cap f^j(B_k)=\emptyset$ for any $\sigma=s,u$, any $0\leq i\leq N_k-1$ and any $k\geq 0$.

We take the union of $(B_s(Q),\mathcal{C}_s(Q),\mathcal{D}_s(Q))$, $(B_u(Q),\mathcal{C}_u(Q),\mathcal{D}_u(Q))$ and $(B_0,\mathcal{C}_0,\mathcal{D}_0)$. To simplify the notations, we still denote the union by $(B_0,\mathcal{C}_0,\mathcal{D}_0)$. By Remark 2.33, we know the modified $(B_0,\mathcal{C}_0,\mathcal{D}_0)$ is still a perturbation domain of order N_0 for $(f,\mathcal{U}_0)$. Denote D'_k the union of the compact sets of the family $\mathcal{D}_k$ for each $k\geq 0$. By a similar argument as in [15, Section 4.3], we assume that z is not in any of the perturbation domains that we have choose.

Recall that the support of the perturbation domain B_k is $\operatorname{Supp}(B_k)=\bigcup_{0\leq n\leq N_k-1} f^n(B_k)$. From the above constructions, the supports of the perturbation domains $(B_k)_{k\geq 0}$ are pairwise disjoint and are contained in U. Moreover, we have that $\operatorname{Supp}(B_k)\subset U_{k-1}\setminus\overline{U_{k+2}}$ for any $k\geq 0$.

Lemma 3.20. *There is an infinitely long pseudo-orbit $Y = (y_0, y_1, \cdots)$ for f contained in X that has jumps only in tiles of $(\mathcal{C}_k)_{k\geq 0}$ with $y_0 = z$ and $d(y_n, K) \to 0$ as $n \to \infty$. Moreover, for each $k \geq 0$, there is a minimal number l_k, such that $y_i \in U_k$ for all $i \geq l_k$.*

Proof. By the former constructions, any point $x \in X_0$ has a positive iterate contained in the union of the interior of the compact set D'_0 and the open sets $W(Q)$ for all periodic orbits $Q \subset X \cap Per_{N_0}(f)$. Any point $x \in X_k$ has a positive iterate contained in the union of the interior of compact sets D'_k for $k \geq 1$. By the compactness of the sets X_k, there are integers T_k, compact sets $\tilde{D}_k \subset D'_k$, and compact sets $\tilde{W}(Q) \subset W(Q)$, such that

- all points $x \in X_0$ will enter the union of $\tilde{D}_0$ and $\tilde{W}(Q)$ for all $Q \subset X \cap Per_{N_0}(f)$ in time bounded by T_0,
- all points $x \in X_k$ will enter in $\tilde{D}_k$ for $k \geq 1$ in time bounded by T_k.

We can assume that T_0 is larger than $n_0(Q)$, for any $Q \subset X \cap Per_{N_0}(f)$ (recall that $n_0(Q)$ is obtained from Lemma 3.19).

Setting of the constants. For any $k \geq 0$, set η_k to be smaller than half of the minimum of the distances between a point in the $(\bigcup_{Q\subset X\cap Per_{N_0}(f)} \tilde{W}(Q)) \cup (\bigcup_{0\leq i\leq k} \tilde{D}_i)$ and a point in the completement of the set $(\bigcup_{Q\subset X\cap Per_{N_0}(f)} W(Q)) \cup (\bigcup_{0\leq i\leq k} D'_i)$. Moreover, we also assume that η_k is smaller than half of the minimum of the distances between a point in $f(\overline{M \setminus U_k})$ to a point in U_{k+1}, and smaller than the minimum of the distances between a point in a compact set $D \in \mathcal{D}_k$ and a point on the boundary of the tile $C \in \mathcal{C}_k$ that contains D. Then for any $k \geq 0$, there is a number $0 < \varepsilon_k < \eta_k$, such that for any ε_k-pseudo-orbit $(x_0, \cdots, x_{T_k})$, we have $d(x_i, f^i(x_0)) < \frac{1}{2}\eta_k$, and $d(x_i, f^{i-T_k}(x_{T_k})) < \frac{1}{2}\eta_k$ for all $0 \leq i \leq T_k$. For each ε_k, there is a number $\delta_k \in (0, \frac{1}{3}\varepsilon_k)$, such that, for any two points $x, y \in M$, if $d(x, y) < \delta_k$, then $d(f(x), f(y)) < \frac{1}{3}\varepsilon_k$. Without loss of generality, we can assume that the sequences $(\eta_k)_{k\geq 0}$, $(\varepsilon_k)_{k\geq 0}$ and $(\delta_k)_{k\geq 0}$ are strictly decreasing sequences.

The sets $\tilde{X}_k$ and the pseudo-orbits Z_k. Now we take a finite δ_k-dense set $\tilde{X}_k$ of X_k for any $k \geq 0$, such that $z \in \tilde{X}_0$. For any $k \geq 0$, take a δ_k-pseudo-orbit $(y_1^k, \cdots, y_{m_k}^k)$ in $X \setminus K$, such that $y_1^k = z$ and $d(y_{m_k}^k, K) < \delta_k$. Then we project this pseudo-orbit to the set $\bigcup_{i\geq 0} \tilde{X}_i$: if $y_j^k \in X_i \setminus X_{i+1}$, then there is $z_j^k \in \tilde{X}_i$, such that $d(y_j^k, z_j^k) < \delta_i$. Then the pseudo-orbit $Z_k = (z_1^k, \cdots, z_{m_k}^k)$ is a pseudo-orbit contained in $\bigcup_{i\geq 0} \tilde{X}_i$ that connects z to K, where $z_1^k = z$.

Recall that $(U_k)_{k\geq 0}$ is a sequence of decreasing neighborhoods of K and $X_k = X \cap \overline{(U_k \setminus U_{k+1})}$. Hence, if $y_j^k, y_{j+1}^k \in U_i$, then we have $d(f(z_j^k), z_{j+1}^k) \leq d(f(z_j^k), f(y_j^k)) + d(f(y_j^k), y_{j+1}^k) + d(y_{j+1}^k, z_{j+1}^k) < \frac{1}{3}\varepsilon_i + \delta_k + \delta_i < \frac{2}{3}\varepsilon_i + \frac{1}{3}\varepsilon_k$. Thus

$d(f(z_j^k), z_{j+1}^k) < \varepsilon_i$ when $k \geq i$. For any $k \geq 0$, by cutting some part of Z_k, we can assume that $z_j^k \neq z_l^k$ for any $j \neq l$. Then for any $k \geq 0$, there is a minimal integer $l(m,k)$, such that $z_i^k \in U_m$ for all $i > l(m,k)$.

The infinitely long pseudo-orbit Z. Since $\tilde{X}_k$ is a finite set for any $k \geq 0$, one can extract a subsequence (Z_k^1) of (Z_k), such that all pseudo-orbits in this subsequence have the same piece before staying in U_1, that is to say, $(z_1^k, \cdots, z_{l(1,k)}^k)$ are equal to each other for any $Z_k \in \{Z_k^1\}$. Similarly, there is a subsequence (Z_k^2) of (Z_k^1), such that all pseudo-orbits in this subsequence have the same piece before staying in U_2. We can continue this process, and finally, by taking the limit, we can get an infinitely long pseudo-orbit $Z = (z_1, z_2, \cdots)$ such that $z_1 = z$, $d(z_n, K) \to 0$ as $n \to \infty$. Moreover, if $z_j, z_{j+1} \in X_i$, then $d(f(z_j), z_{j+1}) < \varepsilon_i$, since Z is a limit set of (Z_k).

By the analysis of Lemma 4.6 in [15], the pseudo-orbit $Z = (z_1, z_2, \cdots)$ has the property stated in the following claim. We omit the proof here since it follows exactly the proof of Lemma 4.6 in [15].

Claim 3.21. *There is a strictly increasing sequence $t_0 = 1, t_1, \cdots$, such that for $j > 0$, z_{t_j} is contained in a compact set E_j of $\bigcup_{k\geq 0} \mathcal{D}_k$. Moreover, for any $j \geq 0$,*

- *if $E_j \in \mathcal{D}_0$, then either $t_j - t_{j-1} < T_1$ or there is $Q \subset X \cap Per_{N_0}(f)$, such that $E_{j-1} \in \mathcal{D}_s(Q)$ and $E_j \in \mathcal{D}_u(Q)$,*
- *if $E_j \in \mathcal{D}_k$ for some $k \geq 1$, then $t_j - t_{j-1} < T_k$.*

Construction of the pseudo-orbit Y from Z. Now we replace some part of Z to get an infinitely long pseudo-orbit that connects $\tilde{U}$ to K, accumulates to K in the future, and has jumps only in the tiles of the perturbation domains. Using Claim 3.21, we construct Y as the following.

- If $E_j \in \mathcal{D}_0$ and $t_j - t_{j-1} < T_1$ or if $E_j \in \mathcal{D}_k$ where $k \geq 1$, we replace the piece of pseudo-orbit $(z_{t_{j-1}+1}, \cdots, z_{t_j})$ by the piece of true orbit $(f(z_{t_{j-1}}), f^2(z_{t_{j-1}}), \cdots, f^{t_j - t_{j-1}}(z_{t_{j-1}}))$.
- If $E_j \in \mathcal{D}_0$ and $t_j - t_{j-1} \geq T_1$, we have that there is $Q \subset X \cap Per_{N_0}(f)$, such that $E_{j-1} \in \mathcal{D}_s(Q)$ and $E_j \in \mathcal{D}_u(Q)$. By Lemma 3.19, there is a piece of true orbit $(x, f(x), \cdots, f^t(x))$ such that $x \in E_{j-1}$, $f^t(x) \in E_j$ and $t \leq n_0(Q) < T_0$. Then we replace the piece of pseudo-orbit $(z_{t_{j-1}+1}, \cdots, z_{t_j})$ by the piece of true orbit $(f(x), f^2(x), \cdots, f^t(x))$.

Then we get a new pseudo-orbit $Y = (y_0, y_1, \cdots)$.

The property of the pseudo-orbit Y. From the construction of the pseudo-orbit Y, we can see that two nearby points y_i, y_{i+1} satisfy the following properties.

- If $y_i \notin \mathcal{D}_k$ for any $k \geq 0$, then $f(y_i) = y_{i+1}$.
- If there exists $k \geq 0$, such that $y_i \in \mathcal{D}_k$, then y_i and $f^{-1}(y_{i+1})$ are in a same tile of $\mathcal{C}_k$.

This implies that Y has jumps only in tiles of $(\mathcal{C}_k)_{k\geq 0}$ with $y_0 = z$ and $d(y_n, K) \to 0$ as $n \to \infty$. Moreover, there is a minimal number l_k, such that $y_i \in U_k$ for all $i \geq l_k$ and all $k \geq 0$.

This finishes the proof of Lemma 3.20. □

Remark 3.22. *In Lemma 3.20, we only need to guarantee that the pseudo-orbit Y obtained has jumps only in the tiles of $(\mathcal{C}_k)_{k\geq 0}$. We do not have to consider the scale of jumps at each step.*

3.5.4 The connecting processes

We take the infinitely long pseudo-orbit $Y = (y_0, y_1, \cdots)$ with $y_0 = z$ contained in X from Lemma 3.20. Recall that Y has jumps only in tiles of $(\mathcal{C}_k)_{k\geq 0}$ and $d(y_n, K) \to 0$ as $n \to \infty$. Moreover, for each $k \geq 0$, there is a minimal integer l_k, such that $y_i \in U_k$ for all $i \geq l_k$. We have the following lemma.

Lemma 3.23. *For each $k \geq 0$, there are a diffeomorphism f_k, an infinitely long pseudo-orbit $Y_k = (y_0^k, y_1^k, \cdots)$ of f_k with $y_0^k = z$, and two sequences of positive integers $(m_k)_{k\geq 0}$ and $(n_k)_{k\geq 0}$, such that for any $k \geq 0$, the following properties are satisfied.*

1. *There is $\phi_k \in \mathcal{V}_k$, such that $\phi_k|_{M\setminus \mathrm{Supp}(B_k)} = Id|_{M\setminus \mathrm{Supp}(B_k)}$, and $f_k = f_{k-1} \circ \phi_k$, where we put $f = f_{-1}$.*
2. *The integer m_k is the smallest positive integer, such that $f_{k+1}^{m_k}(z) \in U_k$. Moreover, we have $m_k < m_{k+1}$.*
3. *The piece of pseudo-orbit $(y_0^{k+1}, y_1^{k+1}, \cdots, y_{m_k-1}^{k+1})$ of f_{k+1} coincides with $(z, f_{k+1}(z), \cdots, f_{k+1}^{m_k-1}(z))$.*
4. *$n_k \leq l_{k+2}$, and $y_{n_k+i}^k = y_{l_{k+2}+i}$, for all $i \geq 0$.*
5. *The pseudo-orbit Y_k of f_k has jumps only in the tiles $\{\mathcal{C}_{k+1}, \mathcal{C}_{k+2}, \cdots\}$.*

Proof. We build the sequences by induction. We construct f_{k+1}, Y_{k+1}, n_{k+1} and m_k after f_k, Y_k, n_k and m_{k-1} has been built.

The constructions of f_0, Y_0, n_0: the case $n = 0$. Recall that the pseudo-orbit Y of f has jumps only in tiles of $(\mathcal{C}_k)_{k\geq 0}$ and l_k is the minimal integer, such that $y_i \in U_k$ for all $i \geq l_k$.

We consider the finite pseudo-orbit $(y_0, y_1, \cdots, y_{l_2})$, which also has jumps only in titles of $(\mathcal{C}_k)_{k\geq 0}$ since it is a piece of Y. Moreover, we have that $y_0 = z$ and $y_0, y_{l_k} \notin \mathrm{Supp}(B_0)$ by the former constructions. By Lemma 2.33, there are a diffeomorphism $f_0 \in \mathcal{U}_0$, a positive integer n_0 and a new pseudo-orbit $Y_0^0 = (\hat{y}_0^0, \hat{y}_1^0, \cdots, \hat{y}_{n_0}^0)$ of f_0, satisfying the following three properties.

- $\hat{y}_0^0 = y_0 = z$ and $\hat{y}_{n_0}^0 = y_{l_2}$.
- The diffeomorphism f_0 coincides with f outside $\mathrm{Supp}(B_0)$, hence there is $\phi_0 \in \mathcal{V}_0$, such that $\phi_0|_{M\setminus\mathrm{Supp}(B_0)} = Id|_{M\setminus\mathrm{Supp}(B_0)}$, and $f_0 = f \circ \phi_0$, which is the item 1.
- The pseudo-orbit Y_0^0 has only jumps in the tiles $\{\mathcal{C}_1, \mathcal{C}_2, \cdots\}$.
- $n_0 \leq l_2$.

Now we consider the infinitely long pseudo-orbit $Y_0 = (y_0^0, y_1^0, \cdots)$ of f_0 which is a composition of Y_0^0 and $(y_{l_2}, y_{l_2+1}\cdots)$. That is to say $y_i^0 = \hat{y}_i^0$ when $0 \leq i \leq n_0$ and $y_i^0 = y_{l_2+i-n_0}$ when $i > n_0$. Then the diffeomorphism f_0, the pseudo-orbit Y_0 and the integer n_0 satisfy the following properties.

- Y_0 has jumps only in the tiles $\{\mathcal{C}_1, \mathcal{C}_2, \cdots\}$, which is the item 5.
- $n_0 \leq l_2$ and $y_{n_0+i}^0 = y_{l_2+i}$ for any $i \geq 0$, which is the item 4.

Notice that we do not have to check the items 2 and 3 for the case $k = 0$.

The constructions f_{k+1}, Y_{k+1}, n_{k+1} and m_k: the case $n = k + 1$. We assume that f_k, Y_k, m_{k-1} and n_k have been built. Then we have that the infinitely long pseudo-orbit $Y_k = (y_0^k, y_1^k, \cdots)$ of f_k with $y_0^k = z$, has only jumps in the tiles $\{\mathcal{C}_{k+1}, \mathcal{C}_{k+2}, \cdots\}$. Moreover, the piece of the pseudo-orbit $(y_0^k, y_1^k, \cdots, y_{m_{k-1}-1}^k)$ coincides with $(z, f_k(z), \cdots, f_k^{m_{k-1}-1}(z))$ and the piece of the pseudo-orbit $(y_{n_k}^k, y_{n_k+1}^k, \cdots)$ coincides with $(y_{l_{k+2}}, y_{l_{k+2}+1}, \cdots)$.

Similarly to the construction in the case $k = 0$, we consider the finite pseudo-orbit $(y_0^k, y_1^k, \cdots, y_{n_k}^k, \cdots, y_{n_k+l_{k+3}-l_{k+2}}^k)$ of f_k which also has only jumps in the tiles $\{\mathcal{C}_{k+1}, \mathcal{C}_{k+2}, \cdots\}$ since it is a piece of Y_k. Notice that $y_{n_k+l_{k+3}-l_{k+2}}^k = y_{l_{k+3}}$ and $\mathrm{Supp}(B_{k+1}) \cap \overline{U_{k+3}} = \emptyset$. By Lemma 2.33, there are a diffeomorphism f_{k+1}, a positive integer n_{k+1} and a new pseudo-orbit $Y_{k+1}^0 = (\hat{y}_0^{k+1}, \hat{y}_1^{k+1}, \cdots, \hat{y}_{n_{k+1}}^{k+1})$ of f_{k+1}, satisfying the following three properties.

- $\hat{y}_0^{k+1} = z$ and $\hat{y}_{n_{k+1}}^{k+1} = y_{l_{k+3}}$.

- The diffeomorphism f_{k+1} coincides with f_k outside $\mathrm{Supp}(B_{k+1})$, hence there is $\phi_{k+1} \in \mathcal{V}_{k+1}$, such that $\phi_{k+1}|_{M\setminus \mathrm{Supp}(B_{k+1})} = Id|_{M\setminus \mathrm{Supp}(B_{k+1})}$, and $f_{k+1} = f_k \circ \phi_{k+1}$, which is the item 1.
- The pseudo-orbit Y^0_{k+1} has only jumps in the tiles $\{\mathcal{C}_{k+2}, \mathcal{C}_2, \cdots\}$.
- $n_{k+1} \leq l_{k+3}$.

Similarly to the case when $k = 0$, we consider the infinitely long pseudo-orbit $Y_{k+1} = (y^{k+1}_0, y^{k+1}_1, \cdots)$ of f_{k+1} which is a composition of Y^0_{k+1} and $(y_{l_{k+3}}, y_{l_{k+3}+1} \cdots)$. That is to say $y^{k+1}_i = \hat{y}^{k+1}_i$ when $0 \leq i \leq n_{k+1}$ and $y^{k+1}_i = y_{l_{k+3}+i-n_{k+1}}$ when $i > n_{k+1}$. Then the diffeomorphism f_{k+1}, the pseudo-orbit Y_{k+1} and the integer n_{k+1} satisfy the following properties.

- Y_{k+1} has jumps only in the tiles $\{\mathcal{C}_{k+2}, \mathcal{C}_{k+3}, \cdots\}$, which is the item 5.
- $n_{k+1} \leq l_{k+3}$ and $y^0_{n_{k+1}+i} = y_{l_{k+3}+i}$ for any $i \geq 0$, which is the item 4.

Then we take the smallest integer m_k, such that $f^{m_{k+1}}_{k+1}(z) \in U_{k+1}$. To be precise, we take m_k in the following way:

- we take $m_0 = 1$,
- when $k \geq 1$, we take m_k such that $f^{m_{k+1}}_{k+1}(z) \in U_{k+1}$, and for all $0 \leq i < m + k + 1$, we have $f^i_{k+1}(z) \notin U_k$.

Since $\mathrm{Supp}(B_{k+1}) \subset U_k \setminus \overline{U_{k+3}}$, the diffeomorphism f_{k+1} coincides with f_k on the piece of orbit $(z, f_k(z), \cdots, f^{m_{k-1}-1}_k(z))$. By the item 2 of Lemma 3.16, we have that $m_{k-1} < m_k$, which is the property 2. The property 3 is satisfied by the choice of m_k. This finishes the proof of Lemma 3.23.

□

End of the proof of Proposition 3.6. Now we consider the sequences $(f_k)_{k\geq 0}$, $(Y_k)_{k\geq 0}$, $(m_k)_{k\geq 0}$ and $(n_k)_{k\geq 0}$ from Lemma 3.23. Recall that $\mathcal{U}_k = f \circ \mathcal{V}_k$ where $\mathcal{V}_k$ satisfies the property (F) in Definition 2.24. Then the sequence of diffeomorphism $f_k = f \circ \phi_0 \circ \cdots \circ \phi_k$ converges to a diffeomorphism $g \in \mathcal{U}$. And since the diameters of the pairwise disjoint perturbations domains can be chosen arbitrarily small by Lemma 2.36, we can take g to be arbitrarily C^0-close to f. Moreover, since the supports of all perturbation domains of $(B_k)_{k\leq 0}$ are contained in U, we have that $g = f|_{M\setminus U}$.

Since $\mathrm{Supp}(B_{k+1}) \subset U_k \setminus \overline{U_{k+3}}$, by the items 2 and 3 of Lemma 3.23, the piece of orbit $(z, f_k(z), \cdots, f^{m_{k-1}-1}_k(z))$ is also a piece of orbit of f_n, when $n \geq k + 1$. This implies that the limit of the sequence of pseudo-orbits Y_k is the positive orbit of z under g since the sequence $(m_k)_{k\geq 0}$ is strictly increasing. By the item 4 of Lemma 3.23, we can see that $\mathrm{Orb}^+(z, g)$ has only finitely many points outside U_k for any $k \geq 0$ (bounded by n_k), hence $\omega(z, g) \subset K$. This finishes the proof of Proposition 3.6. □

Chapter 4

Weak periodic orbits inside non-hyperbolic homoclinic classes

In this chapter, we give the proof of Theorem A and some applications in Section 4.4 and Section 4.5. To make it convenient, we state again Theorem A.

Theorem A. *For generic $f \in \mathrm{Diff}^1(M)$, if a homoclinic class $H(p)$ of f is not hyperbolic, then $H(p)$ contains weak periodic orbits: there exists a sequence of periodic orbits homoclinically related to p that have a Lyapunov exponent converging to 0.*

4.1 A more general version of Theorem A

In this section, we give a more general result than Theorem A.

Theorem D. *For generic $f \in \mathrm{Diff}^1(M)$, assume that p is a hyperbolic periodic point of f. If the homoclinic class $H(p)$ has a dominated splitting $T_{H(p)}M = E \oplus F$, with $\dim E \leq \mathrm{Ind}(p)$, such that the bundle E is not contracted, then there are periodic orbits in $H(p)$ with index $\dim(E)$ that have the maximal Lyapunov exponents along E arbitrarily close to 0.*

Remark 4.1. *In the assumption of Theorem D, if $\dim E = \mathrm{Ind}(p)$, then the weak periodic orbits obtained have the same index as p. Thus by the genericity assumption, they are homoclinically related with $\mathrm{Orb}(p)$.*

We give an explanation how Theorem D implies Theorem A. We assume that all Lyapunov exponents of periodic orbits that are homoclinically related to $\mathrm{Orb}(p)$ are uniformly away from 0. Then by the genericity assumption, $H(p)$ has a dominated splitting $T_{H(p)}M = E \oplus F$, with $\dim E = \mathrm{Ind}(p)$, (see [45] and Proposition 4.8 of [17]). By the conclusion of Theorem D and the assumption of no existence of weak periodic orbits homoclinically related to $\mathrm{Orb}(p)$, we get that the bundle E is contracted. With the same argument for f^{-1}, we get that the bundle F is expanded for f. Hence $T_{H(p)}M = E \oplus F$ is a hyperbolic splitting and we get the conclusion of Theorem A.

4.2 Norm of products and product of norms: reduction of the proof of Theorem D

Theorem D essentially follows from the theorem below.

Theorem E. *For generic $f \in \mathrm{Diff}^1(M)$, assume that p is a hyperbolic periodic point of f and that the homoclinic class $H(p)$ has a dominated splitting $T_{H(p)}M = E \oplus F$, with $\dim E \leq \mathrm{Ind}(p)$, such that the bundle E is not contracted. Then there are a constant $\lambda_0 \in (0,1)$, an integer $m_0 \in \mathbb{N}$, satisfying: for any $m \in \mathbb{N}$ with $m \geq m_0$, any constants $\lambda_1, \lambda_2 \in (\lambda_0, 1)$ with $\lambda_1 < \lambda_2$, there is a sequence of different periodic orbits $\mathcal{O}_k = \mathrm{Orb}(q_k)$ with period $\tau(q_k)$ contained in $H(P)$, such that*

$$\lambda_1{}^{\tau(q_k)} < \prod_{0 \leq i < \tau(q_k)/m} \|Df^m|_{E(f^{im}(q_k))}\| < \lambda_2{}^{\tau(q_k)}.$$

From Theorem E, we can get periodic orbits that have certain controls of the product of norms along the bundle E. To control Lyapunov exponents of the periodic orbits, we have to control the norm of products along the bundle E. We have to use the following perturbation lemma for matrixes to control exponents, see [35, 61] (also see [52, 54]).

Lemma 4.2. *For any integer $n \geq 1$, $K \geq 1$, any constant $\varepsilon > 0$ and $\lambda > 0$, there are two integers N and τ_0, such that for any $A_1, \cdots, A_\tau$ in $GL(n, \mathbb{R})$ with $\tau \geq \tau_0$, and $max_{1 \leq i \leq \tau}\{\|A_i\|, \|A_i^{-1}\|\} \leq K$, if*

$$\prod_{0 \leq i < \tau/N} \|A_{(i+1)N} \cdots A_{iN+2} A_{iN+1}\| \geq \lambda^\tau,$$

then, there are $B_1, \cdots, B_\tau$ in $GL(n, \mathbb{R})$, with $\|B_i - A_i\| < \varepsilon$ and $\|B_i^{-1} - A_i^{-1}\| < \varepsilon$, for all $i = 1, \cdots, \tau$, such that the maximal norm of eigenvalue of $B_\tau \circ \cdots \circ B_2 \circ B_1$ is bigger than λ.

Remark 4.3. *In [35], it is presented for the constant $\lambda = 1$. If $\lambda \neq 1$, then by considering $A_i' = \lambda^{-1} Id \circ A_i$ and applying the special case for the constant 1, we can get the general statement as above.*

Corollary 4.4. *For any integer $n \geq 1$, $K \geq 1$, any constant $\varepsilon > 0$ and $\lambda_1 < \lambda_2$, there are two integers N and τ_0, such that for any $A_1, \cdots, A_\tau$ in $GL(n, \mathbb{R})$ with $\tau \geq \tau_0$, and $max_{1 \leq i \leq \tau}\{\|A_i\|, \|A_i^{-1}\|\} \leq K$, if*

$$\lambda_1^\tau < \prod_{0 \leq i < \tau/N} \|A_{(i+1)N} \cdots A_{iN+2} A_{iN+1}\| < \lambda_2^\tau,$$

then, there are $B_1, \cdots, B_\tau$ in $GL(n,\mathbb{R})$, with $\|B_i - A_i\| < \varepsilon$ and $\|B_i^{-1} - A_i^{-1}\| < \varepsilon$, for all $i = 1, \cdots, \tau$, such that the maximal norm of eigenvalue of $B_\tau \circ \cdots \circ B_2 \circ B_1$ is in the interval (λ_1, λ_2).

Proof. We take ε small enough such that, for any $A \in GL(d,\mathbb{R})$, if $\|A^{-1}\| \le K$, then $B(A,\varepsilon) \in GL(d,\mathbb{R})$, where $B(A,\varepsilon)$ is the ε ball of A. By the assumption of A_i, we have that the maximal norm of eigenvalue of $A_\tau \circ \cdots \circ A_2 \circ A_1$ is smaller than λ_2. By Lemma 4.2, we can get $B_1^0, \cdots, B_\tau^0$ in $GL(d,\mathbb{R})$ that satisfies the conclusion for the number λ_1. We take a path $A_{i,t}|_{0\le t\le 1}$ contained in $B(A_i,\varepsilon)$ that connects A_i to B_i^0. We have that the maximal norm of eigenvalue of $B_\tau^0 \circ \cdots \circ B_2^0 \circ B_1^0$ is bigger than λ_1. Then there must be a time $0 < t < 1$, such that the maximal norm of eigenvalue of $A_{\tau,t} \circ \cdots \circ A_{2,t} \circ A_{1,t}$ is in the interval (λ_1, λ_2). We take $B_i = A_{i,t}$ and get the conclusion. □

Now we give the proof of Theorem D from Theorem E.

Proof. By Theorem E, we get two constants $\lambda_0 \in (0,1)$ and $m_0 \in \mathbb{N}$. We prove that for any $\lambda_0 < \lambda_1 < \lambda_2 < 1$ and any $\varepsilon > 0$, there is a diffeomorphism g that is ε-C^1 close to f and g has a periodic orbit $\mathrm{Orb}(q)$ homoclinic related to p_g such that the largest Lyapunov exponent along E of $\mathrm{Orb}(q)$ is in the interval $(\log\lambda_1, \log\lambda_2)$. Then by the genericity assumption and Lemma 2.1 of [45], f itself has such periodic orbits. Since λ_1 can be taken arbitrarily close to 1, we get the conclusion of the Theorem D.

Take $d = \dim(M)$ and $K = \max\{\|Df\|, \|Df^{-1}\|\}$. Now we fix the constants $\varepsilon > 0$ and $\lambda_1 < \lambda_2$ in $(\lambda_0, 1)$. Since $E \oplus F$ is a dominated splitting, the two bundles E and F are transverse with each other, thus the angle between E and F has a lower bound. As a result, the perturbation of f along the periodic orbit $\mathrm{Orb}(q)$ can be realized by the perturbation restricted to the derivative of f along the two bundles E and F. That is to say, for the constant $\varepsilon > 0$, there is $\varepsilon' > 0$, such that any ε' perturbation of Df on the bundles E and F independently gives an ε perturbation of f. For $\varepsilon' > 0$, we get two integers N and τ_0 by Corollary 4.4.

By Theorem E, there is a periodic orbit $\mathrm{Orb}(q)$ of f with period $\tau > \tau_0$ that is homoclinically related to $\mathrm{Orb}(p)$, such that,

$$\lambda_1{}^\tau < \prod_{0\le i<\tau/m} \|Df^m|_{E(f^{im}(q))}\| < \lambda_2{}^\tau,$$

where $m > m_0$ is a multiple of N. Denote $A_i = Df|_{f^i(q)}$ for $i = 0, \cdots, \tau-1$. Since $E \oplus F$ is a dominated splitting, the two bundles E and F are transverse with each other, thus there is a lower bound of the angle between E and F. By Corollary 4.4, there are $B_0, \cdots, B_{\tau-1}$ in $GL(d,\mathbb{R})$, with $\|B_i - A_i\| < \varepsilon$ and $\|B_i^{-1} - A_i^{-1}\| < \varepsilon$, for all $i = 0, \cdots, \tau-1$, such that, B_i coincides with A_i along the bundle F and the maximal norm of eigenvalue of $B_{\tau-1} \circ \cdots \circ B_1 \circ B_0$ along the bundle E is in the interval $(\lambda_1^\tau, \lambda_2^\tau)$. We take a path $A_{i,t}|_{0\le t\le 1}$ contained in $B(A_i, \varepsilon)$ that connects A_i to B_i such that $A_{i,t}$ coincides with A_i along the bundle F for all $i = 0, \cdots, \tau - 1$ and all $t \in (0, 1)$. If there is a time $t \in (0, 1)$ such that $A_{\tau-1,t} \circ \cdots A_{0,t}$ is not hyperbolic, then there must be a time $t_0 < t$, such that $A_{\tau-1,s} \circ \cdots A_{0,s}$ is hyperbolic for all $0 \leq s \leq t_0$, and the maximal norm of eigenvalue of $A_{\tau-1,t_0} \circ \cdots A_{0,t_0}$ along the bundle E is in the interval $(\lambda_1^\tau, \lambda_2^\tau)$. Otherwise, we can take $t_0 = 1$

Take a small constant $\delta > 0$, since $\mathrm{Orb}(q)$ is homoclinically related to $\mathrm{Orb}(p)$, there exist two points $x \in W^s_\delta(\mathrm{Orb}(q)) \pitchfork W^u(\mathrm{Orb}(p))$ and $y \in W^u_\delta(\mathrm{Orb}(q)) \pitchfork W^s(\mathrm{Orb}(p))$. We take the pair of compact sets $\{x\} \subset W^s_\delta(\mathrm{Orb}(q))$ and $\{y\} \subset W^u_\delta(\mathrm{Orb}(q))$. Then we take a neighborhood V of $\mathrm{Orb}(q)$ such that $x, y \notin V$ and $V \cap (\mathrm{Orb}^-(x) \cup \mathrm{Orb}^+(y)) = \emptyset$. By Lemma 2.42, considering the one-parameter family of linear maps $(A_{i,t})_{i=0,\cdots,\tau-1; t\in[0,t_0]}$, there is a diffeomorphism g that is C^1-ε close to f, such that:

- g coincides with f on $\mathrm{Orb}(q)$ and outside V;
- $x \in W^s_\delta(\mathrm{Orb}(q), g)$ and $y \in W^u_\delta(\mathrm{Orb}(q), g)$;
- $Dg(g^i(q)) = Dg(f^i(q)) = A_{i,t_0}$ for all $i = 0, \cdots, \tau - 1$.

Then we have that $x \in W^s_\delta(\mathrm{Orb}(q), g) \cap W^u(\mathrm{Orb}(p), g)$ and $y \in W^u_\delta(\mathrm{Orb}(q), g) \cap W^s(\mathrm{Orb}(p), g)$, and by another small perturbation if necessary, we can assume that the two intersections are transverse. Then the two periodic orbits $\mathrm{Orb}(q)$ and $\mathrm{Orb}(p)$ of g are still homoclinically related with each other, and the largest Lyapunov exponent of $\mathrm{Orb}(q)$ along the bundle E under the diffeomorphism g is in the interval $(\log \lambda_1, \log \lambda_2)$. This ends the proof of Theorem D. □

4.3 Existence of weak periodic orbits: proof of Theorem E

This section will give the proof of Theorem E. We assume that $\mathcal{R}$ is the residual set of $\mathrm{Diff}^1(M)$ stated in Theorem 2.46 and $f \in \mathcal{R}$ is a diffeomorphism that satisfies the hypothesis of Theorem E. Later we will assume also that f belongs to another two residual subsets $\mathcal{R}_0$ and $\mathcal{R}_1$ defined below.

Since $E \oplus F$ is a dominated splitting and $\dim E \leq \mathrm{Ind}(p)$, we have that: there are $\lambda_0 \in (0, 1)$ and $m_0 \in \mathbb{N}$, such that, for any $m \geq m_0$, the splitting $E \oplus F$ is (m, λ_0^2)-dominated, and, for the hyperbolic periodic orbit $\mathrm{Orb}(p)$,

$$\|Df^{\tau(p)}|_{E(p)}\| < \lambda_0^{\tau(p)},$$

where $\tau(p)$ is the period of $\mathrm{Orb}(p)$. In the following, we fix $m \geq m_0$. In order to simplify the notations, we will assume that $m = 1$ and that p is a fixed point of f, but the general case is identical.

4.3.1 Existence of weak sets

Lemma 4.5. *For any $\lambda \in (\lambda_0, 1)$, there is a λ-E-weak set contained in $H(p)$.*

Proof. Since E is not contracted, the first assumption for the bundle E in Lemma 2.23 is satisfied.

Assume by contradiction that there is a constant $\lambda \in (\lambda_0, 1)$, such that there is no λ-E-weak set contained in $H(p)$. Thus the seconde assumption in Lemma 2.23 is satisfied for the bundle E and the constant λ. Hence, for any $\lambda_1, \lambda_2 \in (\lambda, 1)$ with $\lambda_1 < \lambda_2$, there is a sequence of periodic orbits $\mathrm{Orb}(q_k)$ with period $\tau(q_k)$ that are homoclinically related with each other and that converges to a subset of $H(p)$ such that for any $k \geq 0$, the following properties are satisfied:

$$\lambda_1{}^{\tau(q_k)} \leq \prod_{0 \leq i < \tau(q_k)} \|Df|_{E(f^i(q_k))}\| \leq \lambda_2{}^{\tau(q_k)},$$

Then $H(p) = H(q_k)$ by item 2 of Theorem 2.46, hence $q_k \in H(p)$. It is obvious that $\mathrm{Orb}(q_k)$ is a λ_1-E-weak set contained in $H(p)$, thus is also a λ-E-weak set. This contradicts the assumption that there is no λ-E-weak set contained in $H(p)$. □

4.3.2 Existence of a bi-Pliss point accumulating backward to an E-weak set

From now on, we fix any two numbers $\lambda_1 < \lambda_2$ in $(\lambda_0, 1)$. Then there is a λ_2-E-weak set contained in $H(p)$. By Lemma 2.19, any λ_2-E-weak set K is (C, λ_0, F)-expanded for some constant $C > 0$ depending on K. By [51], any point $x \in K$ has a uniform local unstable manifold $W^u_{loc}(x)$ with a uniform size depending on K.

We extend the dominated splitting $E \oplus F$ to the maximal invariant compact set of a small neighborhood U of $H(p)$ and denote it still by $E \oplus F$. We take a constant $\lambda_3 \in (\lambda_2, 1)$.

Lemma 4.6. *There are a λ_2-E-weak set K, and a λ_3-bi-Pliss point $x \in H(p) \setminus K$ satisfying: $\alpha(x) = K$.*

It is obvious that any compact invariant subset of a λ_2-E-weak set is still a λ_2-E-weak set. So we only have to prove that: *there are a λ_2-E-weak set K, and a λ_3-bi-Pliss point $x \in H(p) \setminus K$ satisfying: $\alpha(x) \subset K$.*

Proof. By Lemma 4.5, there exists a λ_2-E-weak set in $H(p)$. To prove Lemma 4.6, we consider two cases: either all the λ_2-E-weak sets are uniformly E-weak or not. More precisely, if we take the closure of the union of all λ_2-E-weak sets contained in $H(p)$, and denote it by $\hat{K}$, then there are two cases: either $\hat{K}$ is still a λ_2-E-weak set or not.

The uniform case: $\hat{K}$ is a λ_2-E-weak set

In this case, $\hat{K}$ is the maximal λ_2-E-weak set in $H(p)$ and we will take $K = \hat{K}$.

Claim 4.7. *K is locally maximal in $H(p)$.*

Proof. We prove by contradiction. Assume that K is not locally maximal in $H(p)$. Take a decreasing sequence of neighborhoods $(U_n)_{n\geq 0}$ of K, such that $\cap_n U_n = K$. Then for any $n \geq 0$, there is a compact invariant set $K_n \subset U_n \cap H(p)$ such that $K \subsetneqq K_n$. Since K is the maximal λ_2-E-weak set in $H(p)$, we have that K_n is not a λ_2-E-weak set, thus there is a λ_2-E-Pliss point $y_n \in K_n$. Take a converging subsequence of (y_n), and assume y is the limit point. Then we have that $y \in K$ and y is a λ_2-E-Pliss point. This contradicts the fact that K is a λ_2-E-weak set. $\square$

Since K is locally maximal in $H(p)$, there is a neighborhood U of K such that K is the maximal compact invariant set contained in $U \cap H(p)$. Then there is a point $z \in (U \cap H(p)) \setminus K$, such that $\alpha(z) \subset K$.

Claim 4.8. *There exists at least one λ_2-E-Pliss point contained in $\omega(z)$.*

Proof. We proof this claim by absurd. If $\omega(z)$ contains no λ_2-E-Pliss points, by item 1 of Corollary 2.21, $\mathrm{Orb}(z) \cup \omega(z)$ contains no λ_2-E-Pliss points. Then $K \cup \mathrm{Orb}(z) \cup \omega(z)$ is a λ_2-E-weak set, which contradicts the maximality of λ_2-E-weak set K since $z \notin K$. Thus $\omega(z)$ contains at least one λ_2-E-Pliss point. $\square$

Since K is a λ_2-E-weak set, by the domination, for any point $w \in K$, there is an integer n_w, such that $\prod_{i=0}^{n_w-1} \|Df^{-1}|_{F(f^{-i}(w))}\| \leq \left(\frac{\lambda_0^2}{\lambda_2}\right)^{n_w} < {\lambda_0}^{n_w}$. By item 2 of Corollary 2.21, considering the bundle F, there are infinitely many λ_1-F-Pliss points for f^{-1} on $\mathrm{Orb}^-(z)$. We take all the λ_1-F-Pliss points $\{f^{n_i}(z)\}$ with $n_{i+1} > n_i$ on $\mathrm{Orb}(z)$ and consider the following two cases:

- **(a)** either the sequence (n_i) has an upper bound or $(n_{i+1} - n_i)$ can be arbitrarily large;
- **(b)** the sequence (n_i) has no upper bounds and $(n_{i+1} - n_i)$ is bounded.

Claim 4.9. *In case* **(a)***, there exists a λ_2-E-Pliss point $y \in H(p)$, such that, for any $\delta > 0$, there is $n_i \in \mathbb{Z}$, satisfying $d(y, f^{n_i}(z)) < \delta$. Thus, by taking δ small enough, we can take $x \in W^u(f^{n_i}(z)) \cap W^s(y)$, such that x is a λ_3-bi-Pliss point.*

Proof. If the sequence $\{n_i\}$ has an upper bound, we take the maximal n_i. That is to say, $f^{n_i}(z)$ is a λ_1-F-Pliss point for f^{-1}, and, there is no λ_1-F-Pliss point for f^{-1} on $\mathrm{Orb}^+(f^{n_i}(z))$. By item 3 of Lemma 2.22, we have that $f^{n_i}(z)$ is also a λ_1-E-Pliss point, thus $f^{n_i}(z)$ is a λ_1-bi-Pliss point. We take $x = y = f^{n_i}(z)$ in this case.

Otherwise, the sequence $\{n_i\}$ has no upper bounds but $(n_{i+1} - n_i)$ can be arbitrarily large. By item 1 of Lemma 2.22, we can take a subsequence of $\{n_i\}$ such that $f^{n_i}(z)$ converges to a λ_1-bi-Pliss point $y \in \omega(z)$. Then for any $\delta > 0$, we can take n_i large enough, such that $d(y, f^{n_i}(z)) < \delta$, and moreover, we can take $x \in W^u(f^{n_i}z) \cap W^s(y)$, such that, $d(f^j(x), f^j(y)) < \delta$ and $d(f^{-j}(x), f^{-j}(f^{n_i}z)) < \delta$, for all $j \geq 0$. Thus by taking δ small enough, x is a λ_3-bi-Pliss point. ☐

Claim 4.10. *In case* **(b)***, there is a λ_2-E-Pliss point $y \in \omega(z)$, such that, there is $n \in \mathbb{N}$, satisfying $W^u(f^n(z)) \cap W^s(y) \neq \emptyset$. Thus we can take a point $x \in W^u(f^n(z)) \cap W^s(y)$, such that* $\mathrm{Orb}(x)$ *contains some λ_3-bi-Pliss point.*

Proof. In this case, there are infinitely many λ_1-F-Pliss points for f^{-1} on $\mathrm{Orb}^+(z)$, and the time between any consecutive λ_1-F-Pliss points for f^{-1} on $\mathrm{Orb}^+(z)$ is bounded. Then for any point $w \in \overline{\mathrm{Orb}^+(z)}$, there is an integer $n_w \in \mathbb{N}$, such that $\prod_{i=0}^{n_w-1} \|Df^{-1}|_{E(f^{-i}(w))}\| \leq \lambda^{n_w}$. Hence $\overline{\mathrm{Orb}^+(z)}$ is a positive invariant F-expanded compact set, and any point $w \in \overline{\mathrm{Orb}^+(z)}$ has a uniform unstable manifold. By Claim 4.8, there is a λ_2-E-Pliss point $y \in \omega(z)$. For any $\delta > 0$, there is $n \in \mathbb{N}$, such that, $d(y, f^n(z)) < \delta$, and $W^u(f^n(z)) \cap W^s(y) \neq \emptyset$. We take $x \in W^s(y) \cap W^u(f^n(z))$. Then $\alpha(x) = \alpha(z)$ and by item 2 of Corollary 2.21, there are λ_3-F-Pliss points for f^{-1} on $\mathrm{Orb}^-(x)$. Also by taking δ small enough, $d(f^i(x), f^i(y))$ can be small for all $i \geq 0$. Since y is a λ_2-E-Pliss point, we can take x to be a λ_3-E-Pliss point. Then, by item 2 of Lemma 2.22 there exists a λ_3-bi-Pliss point on $\mathrm{Orb}(x)$, we assume that x is such a point. ☐

From the above two claims, we get a λ_3-bi-Pliss point $x \in H(p)$, such that $\alpha(x) \subset K$. We have to show that $x \notin K$. Notice that in the two cases, we both have $\omega(x) = \omega(y)$ where y is a λ_2-E-Pliss point. By item 1 of Corollary 2.21, $\omega(x)$ contains some λ_2-E-Pliss point. Since K contains no λ_2-E-Pliss point, we have that $x \notin K$.

The non-uniform case: $\hat{K}$ is not a λ_2-E-weak set

To prove the non-uniform case, we take a constant $\lambda' \in (\lambda_2, \lambda_3)$. We have the following claim.

Claim 4.11. *For any number $L > 0$ there are a λ_2-E-weak set K and a point $z \in K$, such that z is a λ_1-F-Pliss point for f^{-1}, and, for any $1 \leq n \leq L$, one has*

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i(z))}\| \leq (\lambda')^n.$$

Proof. Since $\hat{K}$ is not a λ_2-E-weak set, there is a λ_2-E- Pliss point in $\hat{K}$. Hence for any number $L > 0$ there is a λ_2-E-weak sets K, and a point $z \in K$, such that, for $1 \leq n \leq L$,

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i(z))}\| \leq (\lambda')^n.$$

We only have to show that we can choose z to be a λ_1-F-Pliss point for f^{-1}. Since K is a λ_2-E-weak set, similarly to the arguments above, by item 2 of Corollary 2.21, there are λ_1-F-Pliss points for f^{-1} on $\mathrm{Orb}^-(z)$. If z is not a λ_1-F-Pliss point for f^{-1}, we can take the minimal number $l \in \mathbb{N}$ such that $w = f^{-l}(z)$ is a λ_1-F-Pliss point for f^{-1}. We claim that for $1 \leq n \leq L$,

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i(w))}\| \leq (\lambda')^n.$$

Hence if we replace z by w, we get the conclusion of the claim. To proof this, we only have to show that for any $1 \leq n \leq l$,

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i(w))}\| \leq \lambda_2^n \leq (\lambda')^n.$$

We prove this by absurd. If the above statement is not true, then there is an integer $k \in \{1, 2, \cdots, l\}$, such that

$$\prod_{i=0}^{k-1} \|Df|_{E(f^i(w))}\| > \lambda_2^k,$$

and for any $1 \leq n < k$,

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i(w))}\| \leq \lambda_2^n.$$

Thus, we have, for any $1 \leq n \leq k$,

$$\prod_{i=1}^{n} \|Df|_{E(f^{k-i}(w))}\| = \left(\prod_{i=0}^{k-1} \|Df|_{E(f^i(w))}\|\right) / \left(\prod_{i=0}^{k-n-1} \|Df|_{E(f^i(w))}\|\right) > \lambda_2^n.$$

By the domination of $E \oplus F$, we have, for all $1 \leq n \leq k$

$$\prod_{i=0}^{n-1} \|Df^{-1}|_{F(f^{k-i}(w))}\| \leq (\frac{\lambda_0^2}{\lambda_2})^n \leq \lambda_1^n.$$

Moreover, since w is a λ_1-F-Pliss point for f^{-1}, we will have, for any $n \geq 1$,

$$\prod_{i=0}^{n-1} \|Df^{-1}|_{F(f^{k-i}(w))}\| \leq \lambda_1^n.$$

Thus $f^k(w) = f^{-l+k}(z)$ is a λ_1-F-Pliss point, contradicting the choice of w. This finishes the proof of Claim 4.11. □

By taking L large enough, the point z in Claim 4.11 is close to a λ_2-E-Pliss point $y \notin K_L$. Since z has a uniform local unstable manifold and y has a uniform local stable manifold, we have that $W^s(y) \cap W^u(z) \neq \emptyset$ if we take these two points close enough. We take the proper L, z and K_L, satisfying this property. Let $K = K_L$. We explain that the λ_2-E-weak set K satisfies Lemma 4.6.

By similar arguments as in the proof of Case **(b)** in the uniform case, we can take a point $\bar{x} \in W^s(y) \cap W^u(z)$ satisfying that $\mathrm{Orb}(\bar{x})$ contains a λ_3-bi-Pliss point $x \in H(p)$. Then we have that $\alpha(x) = \alpha(z) \subset K$. Moreover, since y is a λ_2-E-Pliss point, we know that $\omega(y)$ contains λ_2-E-Pliss points by item 1 of Corollary 2.21. Hence $\omega(x)$ contains λ_2-E-Pliss points because $\omega(x) = \omega(y)$. This implies $x \notin K$ since K is a λ_2-E-weak set. To sum up, we have obtained a λ_2-E-weak set K and a λ_3-bi-Pliss point $x \in H(p) \setminus K$, satisfying that$\alpha(x) \subset K$. This finishes the proof of Lemma 4.6. □

4.3.3 Continuation of Pliss points

Denote by $\mathcal{M}$ the space of all compact subsets of M, associated with the Hausdorff topology. Denote by $\mathcal{S}$ the space of all finite subsets of $M \times \mathcal{M}$ associated with the Hausdorff topology. For any positive integer $N \in \mathbb{N}$, and a diffeomorphism $g \in \mathrm{Diff}^1(M)$, denote by $Per_N(g)$ the set of periodic points of g with period less than or equal to N, and denote by $\mathcal{C}(q, g)$ the chain recurrence class of a periodic point q of g. It is well-known that for any $N \geq 1$, there is a dense and open subset $\mathcal{U}_N \subset \mathrm{Diff}^1(M)$, such that, for any $g \in \mathcal{U}_N$, the set $Per_N(g)$ is a finite set and any point $q \in Per_N(g)$ is a hyperbolic periodic point.

We define a map $\Phi_N : \mathcal{U}_N \mapsto \mathcal{S}$, sending a diffeomorphism g to the set of pairs $(q, P_{\lambda_3}(q, g))$, where $q \in Per_N(g)$, and $P_{\lambda_3}(q, g)$ is a compact set contained in $\mathcal{C}(q, g)$ defined as following:

- If $\mathcal{C}(q, g)$ has a λ_0^2-dominated splitting $E \oplus F$ such that $\dim(E) = \mathrm{Ind}(q)$, then the set $P_{\lambda_3}(q, g)$ is the set of λ_3-E-Pliss points contained in $\mathcal{C}(q, g)$.
- Otherwise, $P_{\lambda_3}(q, g) = \emptyset$.

Lemma 4.12. *For each positive integer $N \in \mathbb{N}$, the set of continuity points of Φ_N, denoted by $\mathcal{B}_N$, is a residual subset of* $\mathrm{Diff}^1(M)$.

Proof. Assume $g \in \mathrm{Diff}^1(M)$ and p_g is a hyperbolic periodic point of g. There is a C^1-neighborhood $\mathcal{U}$ of g, such that, for any $h \in \mathcal{U}$, the point p_g has a continuation p_h. For any neighborhood V of $\mathcal{C}(q, g)$, there is a C^1-neighborhood $\mathcal{U}_1 \subset \mathcal{U}$ of g, such that $\mathcal{C}(q, h) \subset V$ for any $h \in \mathcal{U}_1$.

If $\mathcal{C}(q,g)$ has a λ_0^2-dominated splitting, then it is a robust λ_0^2-dominated splitting. More precisely, there is a C^1-neighborhood $\mathcal{U}_2 \subset \mathcal{U}$ of g, such that $\mathcal{C}(q_h,h)$ has a λ_0^2-dominated splitting for any $h \in \mathcal{U}$. Hence by the choice of $\mathcal{U}_N$, there is an open and dense subset $\mathcal{U}'_N \subset \mathcal{U}_N$, such that, for any $g \in \mathcal{U}'_N$, any $q \in \mathrm{Per}_N(g)$, the chain recurrent class $\mathcal{C}(q,g)$ either has a robust λ_0^2-dominated splitting or has no λ_0^2-dominated splitting robustly. Moreover, if there is a sequence of diffeomorphisms $\{g_n\}_{n\geq 0}$ such that g_n converges to g, and g_n has a λ_3-E-Pliss point $x_n \in \mathcal{C}(q_h,h)$, then, any limit point x of the sequence $\{x_n\}$ is a λ_3-E-Pliss point of g.

By the above arguments, we can see that Φ_N is an upper-semi-continuous map restricted to $\mathcal{U}'_N$. It is known that the set of continuity points of a semi-continuous map is a residual subset. Then $\mathcal{B}_N$ contained a residual subset of $\mathcal{U}'_N$. Since $\mathcal{U}'_N$ is open and dense in $\mathcal{U}_N$, we know that $\mathcal{B}_N$ is a residual subset of $\mathcal{U}_N$. Hence $\mathcal{B}_N$ is a residual subset of $\mathrm{Diff}^1(M)$, since $\mathcal{U}_N$ is open and dense in $\mathrm{Diff}^1(M)$. $\square$

Denote by $\mathcal{R}_0 = \cap_{N\geq 1}\mathcal{B}_N$, then $\mathcal{R}_0$ is a residual subset of $\mathrm{Diff}^1(M)$. In the following we take $f \in \mathcal{R}_0 \cap \mathcal{R}$.

4.3.4 The perturbation to make $W^u(p)$ accumulate to K

We take the λ_2-E-weak set $K \subset H(p)$ of f obtained by Lemma 4.6. By Proposition 3.5, one can obtain a heteroclinic orbit connecting p to K by a C^1 perturbation, since $K \subset H(p)$. Hence the set K is still a λ_2-E-weak set if the perturbation is C^1 small. Moreover, using the continuation of Pliss points (Section 4.3.2 and 4.3.3), we can guarantee that the set K is contained in the chain recurrence class of p after the perturbation.

Lemma 4.13. *Assume $f \in \mathcal{R}_0 \cap \mathcal{R}$, then for any neighborhood $\mathcal{U}$ of f in* $\mathrm{Diff}^1(M)$*, there are a diffeomorphism $g_1 \in \mathcal{U}$ and a point $y \in M$, such that,*

- (1) *g_1 coincides with f on the set $K \cup \mathrm{Orb}(p)$, and $y \in W^u(p,g_1)$,*
- (2) *$\omega(y,g_1) \subset K$,*
- (3) *K is contained in $\mathcal{C}(p,g_1)$.*

Proof. By Lemma 4.6, we obtain that, for the diffeomorphism f, there is a λ_3-bi-Pliss point $x \in H(p) \setminus K$ satisfying: $\alpha(x) = K$. Since $K \subset H(p)$, we have that $K \subset \overline{W^u(p)}$. By Proposition 3.5, for any neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, there are a point $y \in W^u(p,f)$ and a diffeomorphism $g_1 \in \mathcal{U}$, such that $\omega(y,g_1) \subset K$, and $y \in W^u(p,g_1)$. Moreover, the diffeomorphism g_1 coincides with f on the set $\mathrm{Orb}^-(x) \cup K \cup \mathrm{Orb}(p)$ and Dg_1 coincides with Df on $\mathrm{Orb}^-(x)$. Thus items (1) and (2) are satisfied, and x is a λ_3-F-Pliss point for g_1^{-1}.

Since p is a hyperbolic fixed point and and $x \in P_{\lambda_3}(p, f)$, by Lemma 4.12 and the fact that f is a continuity point of Φ_1, if we choose g_1 close enough to f (by taking the neighborhood $\mathcal{U}$ small), then there is a λ_3-E-Pliss x' close to x, such that $x' \in \mathcal{C}(p, g_1)$. Moreover, if x' is close enough to x (by taking g_1 close to f), then $W^u(x, g_1) \cap W^s(x', g_1) \neq \emptyset$. Hence $K \subset \mathcal{C}(p, g_1)$. This finishes the proof of Lemma 4.13. $\square$

4.3.5 The perturbations to connect p and K by true orbits

In this subsection, we prove that we can get heteroclinic connections between the hyperbolic fixed point p and the weak set K for a diffeomorphism C^1 close to f. In the former subsection, we have got a diffeomorphism g_1 that is C^1 close to f, and an orbit $\mathrm{Orb}(y)$ that connects p to K. Moreover K is still contained in the chain recurrence class of p for g_1. We take two steps to get heteroclinic connections between p and K. First, since $K \subset \mathcal{C}(p, g_1)$, by Proposition 3.6, we can connect K by a true orbit to any neighborhood of p by a C^1 small perturbation. Then, by the hyperbolicity of p, we use the uniform connecting lemma to "push" this orbit onto the stable manifold of p. We will see that in these two steps, the orbit $\mathrm{Orb}(y)$ that connects p to K is not changed.

Lemma 4.14. *Assume $f \in \mathcal{R}_0 \cap \mathcal{R}$, then for any neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, there are a diffeomorphism $g_2 \in \mathcal{U}$ and two points $y, y' \in M$, such that,*

- (1) $y \in W^u(p, g_2)$ *and* $\omega(y, g_2) \subset K$,
- (2) $y' \in W^s(p, g_2)$ *and* $\alpha(y', g_2) \subset \omega(y, g_2)$,
- (3) g_2 *coincides with* f *on the set* $\omega(y, g_2) \cup \mathrm{Orb}(p)$.

Proof. We take several steps to prove the lemma.

Choice of neighborhoods. For any any neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, there are a neighborhood $\mathcal{U}_1 \subset \mathcal{U}$ and three numbers $\rho > 1$, $\delta_0 > 0$ and $N \in \mathbb{N}$ that satisfy the uniform connecting lemma (Theorem 2.26). And we can assume that the fixed point p has a continuation for any $g \in \mathcal{U}_1$. For the neighborhood $\mathcal{U}_1$, there are a smaller neighborhood $\mathcal{U}' \subset \mathcal{U}_1$ of f and an integer T satisfying the conclusions of Proposition 3.6. By the hyperbolicity of periodic orbits of f, for the integer T, there is a neighborhood $\mathcal{U}_2 \subset \mathrm{Diff}^1(M)$ of f, such that, for any diffeomorphism $h \in \mathcal{U}_2$, any periodic point of h with period less than or equal to T is hyperbolic. Take a neighborhood $\mathcal{U}_3$ of f in $\mathrm{Diff}^1(M)$, such that $\overline{\mathcal{U}_3} \subset \mathcal{U}_2 \cap \mathcal{U}'$.

The connection from K to a neighborhood of p by pseudo-orbits. By Lemma 4.13, there are a diffeomorphism $g_1 \in \mathcal{U}_3$ and a point $y \in M$, such that:

- g_1 coincides with f on the set $K \cup \mathrm{Orb}(p) \cup \mathrm{Orb}^-(y)$,
- $y \in W^u(p, g_1)$ and $\omega(y, g_1) \subset K \subset \mathcal{C}(p, g_1)$.

Denote $K_0 = \omega(y, g_1)$.

Claim 4.15. *For any neighborhood V of p, there are a g_1 negative invariant compact set X and a point $z \in V \cap X$, satisfying that*

- *the point $p \notin X$,*
- *for any $\varepsilon > 0$, there is a g_1-ε-pseudo-orbit $Y_\varepsilon = (y_0, \cdots, y_m)$ contained in X such that $y_0 \in K_0$ and $y_m = z$.*

Proof. For any neighborhood V of p, take a smaller neighborhood V_0 of p, such that $\overline{V_0} \subset V$. For any $k \geq 1$, there is a g_1-$\frac{1}{k}$-pseudo-orbit $X_k = \{x_0^k, x_1^k, \cdots, x_{m_k}^k\}$, such that, $X_k \cap K_0 = \{x_0^k\}$, and $X_k \cap V_0 = \{x_{m_k}^k\}$. Take a subsequence of $\{X_k\}_{k\geq 1}$ if necessary, we assume X_k converges to a compact set X and $x_{m_k}^k$ converges to a point $z \in \overline{V_0} \subset V$ as k goes to $+\infty$. Obviously, X is a g_1-negative-invariant set, $p \notin X$ and $X \cap K_0 \neq \emptyset$.

Now we prove that for any $\varepsilon > 0$, there is a g_1-ε-pseudo-orbit contained in X from K_0 to z. By the continuity of g_1, for any $\varepsilon > 0$, there is $k > \frac{3}{\varepsilon}$, such that for all $x, y \in M$, if $d(x, y) < \frac{1}{k}$, then $d(g_1(x), g_1(y)) < \frac{\varepsilon}{3}$. Then we take a $\frac{1}{k'}$-pseudo-orbit $X_{k'} = \{x_0^{k'}, x_1^{k'}, \cdots, x_{m_k}^{k'}\}$, such that $x_0^{k'} \in K_0$ and $x_{m_{k'}}^{k'} \in V_0$ for a number $k' > k$. By choosing k' large enough, we can assume that $d_H(X_{k'}, X) < \frac{1}{k}$ and there is a point $y_0 \in X \cap K_0$ such that $d(y_0, x_0^{k'}) < \frac{1}{k}$ and $d(z, x_{m_{k'}}^{k'}) < \frac{1}{k}$. By the assumption, for any $1 \leq i \leq m_{k'} - 1$, there is $y_i \in X$, such that $d(x_i^{k'}, y_i) < \eta$. Denote $Y_\varepsilon = (y_0, \cdots, y_{m_{k'}} = z)$, we prove that Y_ε is a ε-pseudo-orbit of g_1. In fact, for any $0 \leq i \leq m_{k'} - 1$,

$$d(g_1(y_i), y_{i+1}) \leq d(g_1(y_i), g_1(x_i^{k'})) + d(g_1(x_i^{k'}), x_{i+1}^{k'}) + d(x_{i+1}^{k'}, y_{i+1}) < \frac{\varepsilon}{3} + \frac{1}{k'} + \frac{1}{k} < \varepsilon.$$

Hence $Y_\varepsilon \subset X$ is a ε-pseudo-orbit of g_1 from the set K_0 to the point z. □

The perturbation to connect K to a neighborhood of p. We take a local stable manifold $W^s_{loc}(p, g_1)$ of p, and take a compact fundamental domain I_{g_1} of $W^s_{loc}(p, g_1)$. Then there is a number $\delta < \delta_0$, such that, for any point $w \in I_{g_1}$, the N balls $(g_1^j(B(w, 2\delta)))_{0\leq j\leq N-1}$ are each of size smaller than δ_0, pairwise disjoint and disjoint with the set $K \cup \mathrm{Orb}(y, g_1) \cup \mathrm{Orb}(p)$. By the compactness of I_{g_1}, there are finite points $w_1, w_2, \cdots, w_L \in I_{g_1}$ such that $(B(w_i, \delta/\rho))_{1\leq i\leq L}$ is a finite open cover of I_{g_1}. There is a number $\eta > 0$ such that, for any diffeomorphism $h \in \mathcal{U}_1$ that is η-C^0 close to g_1, we have that:

- (a) $W^s_{loc}(p_h, h)$ is C^0 close to $W^s_{loc}(p, g_1)$,
- (b) $(B(w_i, \delta/\rho))_{1\le i\le L}$ is still a finite open cover of a fundamental domain I_h of $W^s_{loc}(p_h, h)$
- (c) for any $1 \le i \le L$, the N balls $(h^j(B(w_i, 2\delta))_{0\le j\le N-1}$ are each of size smaller than δ_0, pairwise disjoint and disjoint with the set $K \cup \mathrm{Orb}(y, g_1) \cup \mathrm{Orb}(p, g_1)$.

Since $y \in W^u(p, g_1)$, $p \notin X$ and X is negative invariant, we have that $\mathrm{Orb}(y, g_1) \cap X = \emptyset$. By the choice of g_1, we have that all periodic orbits of g_1 contained in X with period less than or equal to T are hyperbolic. Under all these hypothesis, $(X \setminus K_0) \cap \overline{\mathrm{Orb}(y, g_1)} = \emptyset$, then there is a neighborhood U_0 of $X \setminus K_0$ such that $U_0 \cap \overline{\mathrm{Orb}(y, g_1)} = \emptyset$. By Proposition 3.6, there is a diffeomorphism $h \in \mathcal{U}_1$ which is η-C^0 close to g_1, such that $h = g_1 = f|_{P\cup \mathrm{Orb}(y)\cup K_0}$, and $\alpha(z, h) \subset K_0$. Thus the above items (a), (b) and (c) are satisfied for such a diffeomorphism h.

The perturbation to get a heteroclinic connection between p and K. By the hyperbolicity of the periodic point p, if we take the neighborhood V of p small enough, then the diffeomorphism h and the point z chosen above would satisfy that the negative orbit of z under h intersect $B(w_i, \delta/\rho)$ for some $i \in \{1, 2, \cdots, L\}$. Since $\alpha(z, h) \subset K_0$ and $B(w_i, \delta/\rho) \cap K_0 = \emptyset$, there is a point $w = h^{-t}(z)$ for some integer $t > 0$, such that $\mathrm{Orb}^-(w) \cap B(w_i, \delta/\rho) = \emptyset$ and w has a positive iterate under h contained in $B(w_i, \delta/\rho)$. By the item (b), there is a point $y' \in W^s(p, h)$, such that $\mathrm{Orb}^+(y', h) \cap (\cup_{0\le j\le N-1} h^i(B(w_i, \delta/\rho))) = \emptyset$ and y' has a negative iterate under h contained in $B(w_i, \delta/\rho)$. By Theorem 2.26, there is a diffeomorphism $g_2 \in \mathcal{U}$, such that y' is on the positive iterate of w under g_2. Moreover, $g_2 = g_1$ on the set $K_0 \cup \mathrm{Orb}(y) \cup \mathrm{Orb}(p) \cup \mathrm{Orb}^-(w) \cup \mathrm{Orb}^+(y')$, hence $g_2 = f$ on the set $\mathrm{Orb}(p) \cup K_0$, where $K_0 = \omega(y, g_1) = \omega(y, g_2)$. Thus the three items of the lemma are satisfied for g_2. This finishes the proof of Lemma 4.14. □

4.3.6 Last perturbation to get a weak periodic orbit

The following lemma estimates the average contraction along the bundle E on periodic orbits.

Lemma 4.16. *Assume $f \in \mathcal{R}_0 \cap \mathcal{R}$. Then for any neighborhood $\mathcal{U}$ of f in* $\mathrm{Diff}^1(M)$*, for any integer $L > 0$, any neighborhood U_p of p, there is $g \in \mathcal{U}$, which coincides with f on* $\mathrm{Orb}(p)$*, satisfying that, g has a periodic point $q \in U_p$ with period $\tau > L$ such that,* $\mathrm{Orb}(q)$ *has the λ_0^2-dominated splitting $E \oplus F$, and*

$$\lambda_1{}^{\tau} \le \prod_{0\le i\le \tau-1} \|Dg|_{E(g^i(q))}\| \le \lambda_2{}^{\tau}.$$

Proof. We take several steps to prove the lemma. We take the λ_2-E-weak set $K \subset H(p)$ of f obtained by Lemma 4.6. Take two numbers λ_1' and λ_2', such that $\lambda_1 < \lambda_1' < \lambda_2' < \lambda_2$.

Choice of neighborhoods and constants. There is a neighborhood V of $H(p)$ and a neighborhood $\mathcal{V} \subset \mathrm{Diff}^1(M)$ of f, such that, for any $h \in \mathcal{V}$, the following properties are satisfied.

- The maximal invariant compact set of h in V has a dominated splitting which is a continuation of $E \oplus F$. To simplify the notations, we still denote this domination by $E \oplus F$.
- The fixed point p has a continuation $p_h \in V$ for h, and $\|Dh|_{E(p_h)}\| < \lambda_0$.
- The chain recurrence class $\mathcal{C}(p_h, h)$ of p_h is contained in V.

Moreover, since K is a λ_2-E-weak set for f, there are a neighborhood $U_K \subset V$ of K and a number N_K, such that, for any point z whose orbit is contained in V, if the piece of orbit $(z, f(z), \cdots, f^n(z))$ is contained in $\overline{U_K}$ with $n \geq N_K$, we have:

$$\prod_{0 \leq i \leq n-1} \|Df|_{E(f^i(z))}\| > {\lambda_2}^n.$$

To simplify the proof, we just assume that $N_K = 1$, but the general case is identical.

We can take the neighborhoods $\mathcal{V}$ and U_p small, such that for any diffeomorphism $h \in \mathcal{V}$, the following additional properties are satisfied.

- For any point $z \in U_p$ whose orbit under h is contained in V, we have that $\frac{\lambda_1}{\lambda_1'} < \frac{\|Dh|_{E(z)}\|}{\|Df|_{E(p)}\|} < \frac{\lambda_2}{\lambda_2'}$.
- For any point $z \in U_K$ whose orbit under h is contained in V, we have that $\|Dh|_{E(z)}\| > \lambda_2$.

We can assume more that $\overline{U_K} \cap \overline{U_p} = \emptyset$ and $\overline{U_K} \cup \overline{U_p} \subset V$. And moreover, we can assume that $\overline{\mathcal{U}} \subset \mathcal{V}$.

By Lemma 4.14, there are a diffeomorphism $g_2 \in \mathcal{U}$ and two points $y, y' \in M$, satisfying that:

- $y \in W^u(p, g_2)$ and $\omega(y, g_2) \subset K$,
- $y' \in W^s(p, g_2)$ and $\alpha(y', g_2) \subset \omega(y, g_2)$,
- g_2 coincides with f on the set $\omega(y, g_2) \cup \mathrm{Orb}(p)$.

We denote $K_0 = \omega(y, g_2)$. Since all periodic points of f are hyperbolic and $g_2 = f|_{K_0}$, then by a C^1 small perturbation if necessary, we can assume that K_0 contains no non-hyperbolic periodic point of g_2.

Choice of time. Now we fix the neighborhoods U_p and U_{K_0}. Then there are two integers l and n_0 satisfying the conclusion of Proposition 3.1 for g_2 and the neighborhood $\mathcal{U}$. Then we take $T_{K_0} > L$ large, such that for any $h \in \mathcal{V}$, the inequality

$$m(h)^{l+n_0}{\lambda_2}^{T_{K_0}} > (\lambda_2')^{T_{K_0}+l+n_0}$$

holds.

By the first item of Proposition 3.1, there is a diffeomorphism $h \in \mathcal{U}$, such that

- h coincides with g_2 on $\mathrm{Orb}(p) \cup \mathrm{Orb}^-(y) \cup \mathrm{Orb}^+(y')$ and outside U_K;
- the point y' is on the positive orbit of y under h, with $n_{K_0} = \#(\mathrm{Orb}(y,h) \cap U_{K_0}) \geq T_{K_0}$ and $n_c = \#((\mathrm{Orb}(y,h) \setminus (U_K \cup U_p)) \leq n_0$.

Hence by the choice of T_{K_0} and the neighborhoods, we have that

$$\prod_{h^i(y) \notin U_p} \|Dh|_{E(h^i(x))}\| > (\lambda_2')^{n_{K_0}+n_c}.$$

Claim 4.17. *There is an integer $m > 0$, such that:*

$$(\lambda_1')^{n_{K_0}+n_c+m+l} < \|Df|_{E(p)}\|^{l+m} \cdot \prod_{h^i(y) \notin U_p} \|Dh|_{E(h^i(x))}\| < (\lambda_2')^{n_{K_0}+n_c+m+l}.$$

Proof. We assume that

$$\prod_{h^i(y) \notin U_p} \|Dh|_{E(h^i(x))}\| = \bar{\lambda}^{n_{K_0}+n_c},$$

then $\bar{\lambda} > \lambda_2'$. The inequality in the claim is equivalent to

$$\frac{(n_{K_0}+n_c)\log\frac{\bar{\lambda}}{\lambda_2'}}{\log\frac{\lambda_2'}{\|Df|_{E(p)}\|}} < l+m < \frac{(n_{K_0}+n_c)\log\frac{\bar{\lambda}}{\lambda_1'}}{\log\frac{\lambda_1'}{\|Df|_{E(p)}\|}}.$$

By the choice of T_{K_0} and $n_{K_0} \geq T_{K_0}$, we have that

$$\frac{(n_{K_0}+n_c)\log\frac{\bar{\lambda}}{\lambda_2'}}{\log\frac{\lambda_2'}{\|Df|_{E(p)}\|}} > l.$$

So we only need that

$$\frac{(n_{K_0}+n_c)\log\frac{\bar{\lambda}}{\lambda_1'}}{\log\frac{\lambda_1'}{\|Df|_{E(p)}\|}} - \frac{(n_{K_0}+n_c)\log\frac{\bar{\lambda}}{\lambda_2'}}{\log\frac{\lambda_2'}{\|Df|_{E(p)}\|}} > 1.$$

It is equivalent to

$$\left(\frac{1}{\log\frac{\lambda_1'}{\|Df|_{E(p)}\|}}-\frac{1}{\log\frac{\lambda_2'}{\|Df|_{E(p)}\|}}\right)\log\bar{\lambda}+\frac{\log\lambda_2'}{\log\frac{\lambda_2'}{\|Df|_{E(p)}\|}}-\frac{\log\lambda_1'}{\log\frac{\lambda_1'}{\|Df|_{E(p)}\|}}>\frac{1}{n_{K_0}+n_c}.$$

Since $\bar{\lambda}>\lambda_2'$, and $n_{K_0}>T_{K_0}$, it is sufficient to acquire that

$$\frac{T_{K_0}(\log\lambda_2'-\log\lambda_1')}{\log\frac{\lambda_1'}{\|Df|_{E(p)}\|}}>1.$$

By taking T_{K_0} large enough, the above inequality is satisfied. □

Choice of the diffeomorphism g**.** We take $g=h_m\in\mathcal{U}$ from item 2 of Proposition 3.1, then g has a periodic orbit $O=\mathrm{Orb}(q)$, such that, $O\setminus U_p=(\mathrm{Orb}(y,h)\setminus U_p$, and $\#(O\cap U_p)=l+m$. Hence the period τ of O equals $n_{K_0}+n_c+m+l$. By the choice of the neighborhood $\mathcal{U}$ and the constants, we have

$$\prod_{0\le i\le\tau-1}\|Dg|_{E(g^i(q))}\|=\prod_{g^i(q)\in U_p}\|Dg|_{E(g^i(q))}\|\prod_{h^i(y)\notin U_p}\|Dh|_{E(h^i(x))}\|.$$

By the choice of the neighborhoods $\mathcal{V}$ and U_p, and the constants λ_1' and λ_2', we have that

$$\left(\frac{\lambda_1}{\lambda_1'}\right)^{l+m}\|Df|_{E(p)}\|^{l+m}<\prod_{g^i(q)\in U_p}\|Dg|_{E(g^i(q))}\|<\left(\frac{\lambda_2}{\lambda_2'}\right)^{l+m}\|Df|_{E(p)}\|^{l+m}.$$

Then by the estimation in Claim 4.17, we can see that

$$\lambda_1{}^{\tau}\le\prod_{0\le i\le\tau-1}\|Dg|_{E(g^i(q))}\|\le\lambda_2{}^{\tau}$$

This finishes the proof of Lemma 4.16. □

4.3.7 The genericity argument

In this subsection, we do the genericity argument to get the conclusion of Theorem E, see like [44].

Take a countable basis $(V_n)_{n\ge1}$ of M, and take the countable family $(U_n)_{n\ge1}$, where each U_n is a union of finitely many sets of $(V_n)_{n\ge1}$. Take the countable pairs $(\eta_n,\gamma_n)_{n\ge1}$ of rational numbers contained in $(\lambda_0,1)$ with $\eta_n<\gamma_n$ for each $n\ge1$.

Let $\mathcal{H}_{n,m}$ be the set of C^1 diffeomorphisms h such that, every h_1 in a C^1 neighborhood $\mathcal{V}\subset\mathrm{Diff}^1(M)$ of h has a hyperbolic periodic point $q\in U_n$

satisfying that the hyperbolic splitting $E^s \oplus E^u$ of $\mathrm{Orb}(q, h_1)$ is a λ_0^2-dominated splitting and

$$\eta_m{}^{\tau(q)} < \prod_{0 \le i \le \tau(q)-1} \|Dh_1|_{E^s(h_1^i)}\| < \gamma_m{}^{\tau(q)},$$

where $\tau(q)$ is the period of q. Let $\mathcal{N}_{n,m}$ be the set of C^1 diffeomorphisms h such that every h_1 in a C^1 neighborhood $\mathcal{V} \subset \mathrm{Diff}^1(M)$ of h has no hyperbolic periodic point $q \in U_n$ satisfying that the hyperbolic splitting $E^s \oplus E^u$ of $\mathrm{Orb}(q, h_1)$ is a λ_0^2-dominated splitting and

$$\eta_m{}^{\tau(q)} < \prod_{0 \le i \le \tau(q)-1} \|Dh_1|_{E^s(h_1^i)}\| < \gamma_m{}^{\tau(q)},$$

where $\tau(q)$ is the period of q.

Notice that $\mathcal{N}_{n,m} = \mathrm{Diff}^1(M) \setminus \overline{\mathcal{H}_{n,m}}$. Hence $\mathcal{H}_{n,m} \cup \mathcal{N}_{n,m}$ is C^1 open and dense in $\mathrm{Diff}^1(M)$. Let

$$\mathcal{R}_1 = \bigcap_{n \ge 1, m \ge 1} (\mathcal{H}_{n,m} \cup \mathcal{N}_{n,m}).$$

Then $\mathcal{R}_1$ is a residual subset of $\mathrm{Diff}^1(M)$, and $\mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{R}$ is also a residual subset of $\mathrm{Diff}^1(M)$.

Claim 4.18. *Assume $f \in \mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{R}$. Then for any two numbers $\lambda_1 < \lambda_2 \in (\lambda_0, 1)$, for any neighborhood U_p of* $\mathrm{Orb}(p)$*, and any integer $L > 0$, there is a periodic point $q \in U_p$ with period $\tau > L$ such that* $\mathrm{Orb}(q)$ *has the λ_0^2-dominated splitting $E \oplus F$, and*

$$\lambda_1{}^{\tau} \le \prod_{0 \le i \le \tau-1} \|Df|_{E(f^i(q))}\| \le \lambda_2{}^{\tau}.$$

Proof. We take two rational numbers $\eta_i, \gamma_i \in (\lambda_0, 1)$, such that $\lambda_1 < \eta_i < \gamma_i < \lambda_2$, and take U_j from the countable basis of M, such that $U_j \subset U_p$. Then by Lemma 4.16, there is a diffeomorphism g arbitrarily C^1 close to f, such that g has a periodic point $q \in U_p$ with period $\tau > T$ such that the λ_0^2-dominated splitting $E \oplus F$ is the hyperbolic splitting on $\mathrm{Orb}(q, g)$, and

$$\eta_i{}^{\tau} \le \prod_{0 \le i \le \tau-1} \|Dg|_{E(g^i(q))}\| \le \gamma_i{}^{\tau}.$$

Then $f \notin \mathcal{N}_{j,i}$, thus $f \in \mathcal{H}_{j,i}$ and f satisfies the conclusion of Claim 4.18. □

Claim 4.19. *Theorem E holds for any diffeomorphisms in $\mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{R}$.*

Proof. Assume $f \in \mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{R}$ and f satisfies the assumptions of Theorem E. By Claim 4.18, we get a sequence of periodic orbits $\mathrm{Orb}(q_k)$ of f, such that $q_k \to p$ with $\tau(q_k) \to \infty$, and

$$\lambda_1{}^{\tau(q_k)} \le \prod_{0 \le i \le \tau(q_k)-1} \|Df|_{E(f^i(q_k))}\| \le \lambda_2{}^{\tau(q_k)}.$$

Hence by the λ_0^2-domination of $E \oplus F$, we have that

$$\prod_{0 \leq i \leq \tau(q_k)-1} \|Df^{-1}|_{F(f^{-i}(q_k))}\| \leq {\lambda_2}^{\tau(q_k)}.$$

Then by item 2 of Lemma 2.21 and item 2 of Lemma 2.22, there is a λ_2-bi-Pliss point r_k on $\mathrm{Orb}(q_k)$ for each k. Taking a subsequence if necessary, we assume (r_k) is a converging sequence. Then there is $l > 0$, such that for any $m, n \geq l$, the stable and unstable manifolds of r_m and r_n intersect respectively, since r_k has uniform stable and unstable manifolds. Hence $(\mathrm{Orb}(q_m))_{m \geq l}$ are homoclinically related together, thus $p \in H(q_k)$. By item 2 of Theorem 2.46, we have that $q_k \in H(p)$. This finishes the proof of the claim. □

The proof of Theorem E is now completed.

4.4 Some applications of Theorem E

4.4.1 Structural stability and hyperbolicity

Recall that a diffeomorphism $f \in \mathrm{Diff}^1(M)$ is *structurally stable*, if there is a C^1 neighborhood $\mathcal{U}$ of f, such that, for any $g \in \mathcal{U}$, there is a homeomorphism $\phi : M \to M$, satisfying $\phi \circ f = g \circ \phi$. The orbital structure of a structurally stable diffeomorphism remains unchanged under perturbations. Mañé proved that the chain recurrent set of a structurally stable diffeomorphism is hyperbolic, see [55]. Here we give a local version about this result.

It is known that a hyperbolic periodic point has a continuation. More precisely, for a hyperbolic periodic point p of a diffeomorphism f with period τ, there is a neighborhood U of $\mathrm{Orb}(p)$ and a C^1 neighborhood $\mathcal{U}$ of f, such that, for any $g \in \mathcal{U}$, the maximal invariant compact set of g in U is a unique periodic orbit with period τ and with the same index as p. We denote this *continuation* of p by p_g for such a diffeomorphism g, and denote the homoclinic class (and chain recurrence class resp.) of p_g by $H(p_g)$ (and $C(p_g)$ resp.). Thus we say that a homoclinic class $H(p)$ of a diffeomorphism f is *structurally stable*, if there is a C^1 neighborhood $\mathcal{U}$ of f, such that, for any $g \in \mathcal{U}$, there is a homeomorphism $\phi : H(p) \to H(p_g)$, satisfying $\phi \circ f|_{H(p)} = g \circ \phi|_{H(p)}$, where p_g is the continuation of p. Similarly we can define the structurally stability for a chain recurrence class $C(p)$ of a hyperbolic point. One asks naturally the following question, which can be seen as a “local" version of the stability conjecture.

Question 4. *Assume p is a hyperbolic point for a diffeomorphism, if $H(p)$ (or $C(p)$) is structurally stable, then is it hyperbolic?*

There are many works related to this question, see for example [45, 67, 76, 79]. In [76] and [79], they prove that structural stability implies hyperbolicity for the chain recurrence class and the homoclinic class respectively of a hyperbolic periodic point, under the hypothesis that the diffeomorphism is far away from tangency, or that the stable or the unstable dimension of this periodic point is 1. With the conclusions of Theorem A, we can give a complete answer to Question 4.

Corollary 4.20. *Assume f is a diffeomorphism in $\mathrm{Diff}^1(M)$ and p is a hyperbolic periodic point of f. If the homoclinic class $H(p)$ is structurally stable, then $H(p)$ is hyperbolic. The conclusion is also valid for $C(p)$.*

To prove Corollary 4.20, we use some of the results in [67, 76, 78, 79]. We take two steps: first, we prove that the statement is true for a residual subset of $\mathrm{Diff}^1(M)$, and then we prove it for all diffeomorphisms in $\mathrm{Diff}^1(M)$.

Assume that $H(p)$ is the homoclinic class of a hyperbolic periodic point p of a diffeomorphism $f \in \mathrm{Diff}^1(M)$. We state two properties as follows:

- $(P1)$ There are $m \in \mathbb{N}$, $C > 0$ and $0 < \lambda < 1$, such that $H(p)$ admits an (m, λ)-dominated splitting $T_{H(p)}M = E \oplus F$ with $\dim(E) = \mathrm{Ind}(p)$. And for any periodic point q homoclinically related to p, denote by $\tau(q)$ the period of q, then the followings are satisfied:

$$\prod_{0 \le i < \tau(q)/m} \|Df^m|_{E(f^{im}(q))}\| < C\lambda^{\tau(q)},$$

$$\prod_{0 \le i < \tau(q)/m} \|Df^{-m}|_{F(f^{-im}(q))}\| < C\lambda^{\tau(q)}.$$

- $(P2)$ $H(p)$ is shadowable and every periodic pseudo-orbit can be shadowed by a periodic orbit.

Now we state the following two Lemmas, whose proofs will be omitted.

Lemma 4.21 (Theorem 1.1 of [79])**.** *Assume that f is a diffeomorphism in $\mathrm{Diff}^1(M)$. If a homoclinic class $H(p)$ is structurally stable, then the property $(P1)$ is satisfied for $H(p)$.*

Lemma 4.22 (Proposition 4.1 of [79])**.** *Assume that f is a diffeomorphism in $\mathrm{Diff}^1(M)$ and p is a hyperbolic periodic point. If the two properties $(P1)$ and $(P2)$ are satisfied, then $H(p)$ is hyperbolic.*

Lemma 4.23 (Proposition 2.4 of [78] and Proposition 3.3 of [78])**.** *The conclusions of Lemma 4.21 and Lemma 4.22 are also valid for the chain recurrence class $C(p)$.*

Proof of Corollary 4.20. By Theorem D, for any diffeomorphism f contained in a residual subset $\mathcal{B} \subset \mathrm{Diff}^1(M)$, if the property $(P1)$ is satisfied for a homoclinic class $H(p)$ of f, then $H(p)$ is hyperbolic. We can take $\mathcal{B}$ such that $\mathcal{B} \subset \mathcal{R}$, where $\mathcal{R}$ is the residual subset in Theorem 2.46. Hence by Lemma 4.21, for any diffeomorphism $f \in \mathcal{B}$, if a homoclinic class $H(p)$ is structurally stable, then it is hyperbolic.

Now we assume that f is an arbitrarily diffeomorphism in $\mathrm{Diff}^1(M)$. If a homoclinic class $H(p)$ of f is structurally stable, then there is a C^1 neighborhood $\mathcal{U}$ of f, such that, for any $h \in \mathcal{U}$, there is a homeomorphism $\phi : H(p) \to H(p_h)$, satisfying $\phi \circ f|_{H(p)} = h \circ \phi|_{H(p)}$. Since $\mathcal{B}$ is residual in $\mathrm{Diff}^1(M)$, we can take a diffeomorphism $g \in \mathcal{B} \cap \mathcal{U}$. Therefore, $H(p_g)$ is structurally stable, where $\mathcal{U}$ is the neighborhood of g in the definition of structurally stable. Then $H(p_g)$ is hyperbolic by the argument above, hence $H(p_g)$ satisfies the property $(P2)$. It is easy to see that the property $(P2)$ is unchanged under conjugacy, thus is satisfied by $H(p)$ since $f \in \mathcal{U}$. The property $(P1)$ is satisfied by $H(p)$ by Lemma 4.21. Then by Lemma 4.22, we have that $H(p)$ is hyperbolic. This finishes the proof for homoclinic classes.

For chain recurrence classes of a hyperbolic periodic point, we only have to show that Corollary 4.20 is valid for $f \in \mathcal{B}$, and then with the same argument as above, we can get the conclusion. Assume $f \in \mathcal{B}$ and p is a hyperbolic periodic point of f. By item 2 of Theorem 2.46, $C(p) = H(p)$. By Lemma 4.23, the property $(P1)$ is satisfied for $C(p)$ and hence for $H(p)$. Thus $C(p) = H(p)$ is hyperbolic. This finishes the proof of Corollary 4.20. □

4.4.2 Partial hyperbolicity

Next result is that for a homoclinic class with a dominated splitting of a C^1-generic diffeomorphism, if the dimensions of the two bundles in the splitting satisfy certain hypothesis, then the splitting is a partially hyperbolic splitting (at least one bundle is hyperbolic).

Corollary 4.24. *For generic $f \in \mathrm{Diff}^1(M)$, if a homoclinic class $H(p)$ has a dominated splitting $T_{H(p)}M = E \oplus F$, such that* $\dim(E)$ *is smaller than the smallest index of periodic orbits contained in $H(p)$, then the bundle E is contracted. Symmetrically, if* $\dim(E)$ *is larger than the largest index of periodic orbits contained in $H(p)$, then the bundle F is expanded.*

Proof. We just prove when $\dim(E)$ is smaller than the smallest index of periodic orbits contained in $H(p)$, thus $\dim(E) < \mathrm{Ind}(p)$. Assume E is not contracted, by the conclusion of Theorem A, we can get a sequence of periodic orbits $\mathrm{Orb}(q_n) \subset H(p)$ with arbitrarily long periods such that $\mathrm{Ind}(q_n) = \dim(E)$, which contradicts to the assumption that $\dim(E) < \mathrm{Ind}(q_n)$. □

As another consequence of Theorem A, we can give a proof of Theorem 1.1 (2) in [39] with a different argument. More precisely, we can prove that for a C^1-generic diffeomorphism far from tangency, a homoclinic class has a partially hyperbolic splitting whose center bundle splits into 1-dimensional subbundles, and the Lyapunov exponents of the periodic orbits along each the center subbundle can be arbitrarily close to 0. Denote $\mathcal{HT}$ the set of diffeomorphisms of $\mathrm{Diff}^1(M)$ that exhibit a tangency.

Corollary 4.25 ([39]). *For generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathcal{HT}}$, a homoclinic class $H(p)$ has a partially hyperbolic splitting $T_{H(p)}M = E^s \oplus E_1^c \oplus \cdots \oplus E_k^c \oplus E^u$ such that each of the center subbundles E_i^c is neither contracted nor expanded and $\dim(E_i^c) = 1$, for all $i = 1, \cdots, k$. Moreover, the minimal index of periodic points contained in $H(p)$ is $\dim(E^s)$ or $\dim(E^s) + 1$, and symmetrically, the maximal index of periodic points contained in $H(p)$ is $d - \dim(E^u)$ or $d - \dim(E^u) - 1$. For each $i = 1, \cdots, k$, there exist periodic orbits contained in $H(p)$ with arbitrarily long periods with a Lyapunov exponent along E_i^c arbitrarily close to 0.*

Proof. We assume that f satisfies the properties in Theorem 2.46 and Theorem A. For a homoclinic class $H(p)$ of f, we denote by $j \geq 1$ and $l \leq d - 1$ the smallest and the largest index of periodic point contained in $H(p)$. By item 4 of Theorem 2.46 (Theorem 1 of [3]), for any $j \leq i \leq l$, there are periodic orbits of index i contained in $H(p)$. Moreover, for any $j \leq i \leq l$, the hyperbolic periodic points with index i are dense in $H(p)$. By Theorem A of [73], $H(p)$ admits a dominated splitting with index i. By Remark 2.8, we have that $H(p)$ admits a dominated splitting $T_{H(p)}M = E^{cs} \oplus E_1^c \oplus \cdots \oplus E_n^c \oplus E^{cu}$, where $\dim(E^{cs}) = j$, $n = l - j$, and $\dim(E_i^c) = 1$ for all $1 \leq i \leq n$. Since $H(p)$ contains hyperbolic periodic points with index i for all $j \leq i \leq l$, we can see easily that the central bundle E_i^c is neither contracting nor expanding, for any $1 \leq i \leq n$.

Now we consider whether the two bundles E^{cs} and E^{cu} are hyperbolic. Here E^{cs} is hyperbolic means it is contracting and E^{cu} is hyperbolic means it is expanding.

Case 1: both E^{cs} and E^{cu} are hyperbolic In this case, the splitting $T_{H(p)}M = E^{cs} \oplus E_1^c \oplus \cdots \oplus E_n^c \oplus E^{cu}$ is a partially hyperbolic splitting. We put $E^s = E^{cs}$ and $E^u = E^{cu}$. Then the the smallest and the largest index of periodic orbits contained in $H(p)$ are $\dim(E^s)$ and $d - \dim(E^u)$ respectively.

Case 2: only one bundle of E^{cs} and E^{cu} is hyperbolic We only prove the case where E^{cs} is not contracting and E^{cu} is expanding. The other case is symmetric. We put $E^u = E^{cu}$. Since E^{cs} is not contracting, by Theorem D, there are periodic orbits with arbitrarily long period and index j, whose largest Lyapunov exponent along E^{cs} converges to 0. Moreover, such periodic orbits form a dense set in $H(p)$. Since f is far away from tangency, the other Lyapunov exponents of such periodic orbits along E^{cs} are uniformly controlled by the largest one. Thus by the Franks' Lemma, there are periodic orbits with index $j-1$ in any neighborhood of $H(p)$ by C^1 small perturbations. Then by a genericity argument like in Section 4.3.7, for the diffeomorphism f, $H(p)$ can be accumulated by periodic orbits with index $j-1$. Therefore, by Theorem A of [73] and Remark 2.8, E^{cs} has a dominated splitting $E^{cs} = E^s \oplus E_0^c$ with $\dim(E_0^c) = 1$. By Corollary 4.24, we have that E^s is contracting. Then the splitting $T_{H(p)}M = E^{cs} \oplus E_1^c \oplus \cdots \oplus E_n^c \oplus E^{cu} = E^s \oplus E_0^c \oplus E_1^c \oplus \cdots \oplus E_n^c \oplus E^u$ is a partially hyperbolic splitting with everything center bundle of dimension one. The smallest index of periodic orbits contained in $H(p)$ is $\dim(E^s)+1$ and the largest one is $d - \dim(E^u)$.

Case 3: neither of E^{cs} and E^{cu} is hyperbolic In this case, we have that E^{cs} is not contracting and E^{cu} is not expanding. With similar arguments in Case 2, both the two bundles have a better dominated splitting $E^{cs} = E^s \oplus E_0^c$ and $E^{cu} = E_{n+1}^c \oplus E^u$. Moreover, the bundle E^s is contracting, the bundle E^u is expanding and $\dim(E_0^c) = \dim(E_{n+1}^c) = 1$. Then we have that $T_{H(p)}M = E^{cs} \oplus E_1^c \oplus \cdots \oplus E_n^c \oplus E^{cu} = E^s \oplus E_0^c \oplus E_1^c \oplus \cdots \oplus E_n^c \oplus E_{n+1}^c \oplus E^u$ is a partially hyperbolic splitting with everything center bundle of dimension one. The smallest and largest index of periodic orbits contained in $H(p)$ are $\dim(E^s)+1$ and $d - \dim(E^u) - 1$ respectively.

This ends the proof of Corollary 4.25.

□

4.4.3 Lyapunov stable homoclinic classes

The following results are about C^1-generic Lyapunov stable homoclinic classes. First, for C^1-generic Lyapunov stable homoclinic classes, we can get a similar conclusion of Corollary 4.24 under a weaker hypothesis.

Corollary 4.26. *For generic $f \in \mathrm{Diff}^1(M)$, if a homoclinic class $H(p)$ is Lyapunov stable and has a dominated splitting $T_{H(p)}M = E \oplus F$ such that* $\dim(E)$ *is larger than or equal to the largest index of periodic orbits contained in $H(p)$, then the bundle F is expanded.*

Proof. From Corollary 4.24, we only have to prove the case where $\dim(E)$ equals the largest index of periodic orbits contained in $H(p)$. The idea of the proof follows from [62] and Section 3 of [7]. We just give an explanation here and for more details, the reader should refer to [62] and Section 3 of [7].

Assume $f \in \mathcal{R}$ satisfies Theorem D where $\mathcal{R}$ is the residual set in Theorem 2.46. There is a neighborhood $\mathcal{U}$ of f, such that the items 3, 4 and 6 stated in Theorem 2.46 are satisfied for $(f, H(p), \mathcal{U})$. We can assume that p has the largest index among the periodic points contained in $H(p)$, hence $\dim(E) = \mathrm{Ind}(p)$. Assume that the bundle F is not expanding, then by the conclusion of the Theorem D, we can get a sequence of periodic orbits $\mathrm{Orb}(q_n)$

homoclinically related to $\mathrm{Orb}(p)$ with arbitrarily long period such that the smallest Lyapunov exponent of $\mathrm{Orb}(q_n)$ along the bundle F can be arbitrarily close to 0. By Lemma 2.3 of [45], we can assume that all the eigenvalues of $\|Df\|$ along $\mathrm{Orb}(q_n)$ are real. Then by Theorem 1 of [48] (Theorem 2.5 in [7]) and a proper construction of a path of diffeomorphism (see [7]), there is a diffeomorphism $g \in \mathcal{U}$ and a periodic point q of g with index larger than $\dim(E)$ such that $W^s(q) \cap W^u(p_g) \neq \emptyset$. By C^1 small perturbation, we can assume that $W^s(q)$ intersects $W^u(p_g)$ transversely. This property is persistent under C^1 perturbation, since $\mathrm{Ind}(q) > \mathrm{Ind}(p_g)$. Hence there is a neighborhood $\mathcal{V} \subset \mathcal{U}$ of g, such that for any $h \in \mathcal{V}$, we have $W^s(q_h) \pitchfork W^u(p_h) \neq \emptyset$. Take a diffeomorphism $h \in \mathcal{V} \cap \mathcal{R}$, we have that $H(p_h)$ is Lyapunov stable by the item 6 of Theorem 2.46. Then $W^u(p_h)$ is contained in $H(p_h)$. By the fact $W^s(q_h) \pitchfork W^u(p_h) \neq \emptyset$, we have that $q_h \in H(p_h)$. This contradicts the item 4 of Theorem 2.46 by the choice of $\mathcal{U}$, since $\mathrm{Ind}(q_h) > \mathrm{Ind}(p_h) = \mathrm{Ind}(p)$.

□

With the conclusion of Corollary 4.26, we can give a positive answer to Conjecture 1 for bi-Lyapunov stable homoclinic classes.

Corollary 4.27. *For generic $f \in \mathrm{Diff}^1(M)$, where M is connected, if a homoclinic class $H(p)$ is bi-Lyapunov stable, then we have:*

- *either $H(p)$ is hyperbolic, hence $H(p) = M$ and f is Anosov,*
- *or f can be C^1 approximated by diffeomorphisms that have a heterodimensional cycle.*

Proof. We assume that the second item does not happen. By Theorem 2.46, all periodic orbits contained in $H(p)$ have the same index. By [62], $H(p)$ has a dominated splitting $T_{H(p)}M = E \oplus F$ such that $\dim(E) = \mathrm{Ind}(p)$. By Corollary 4.26, we have that the bundle F is expanded. With the same argument to f^{-1} and the bundle E, we get that E is contracted for f. Hence the splitting $T_{H(p)}M = E \oplus F$ is hyperbolic. Then $H(p)$ is a hyperbolic chain recurrence class by item 2 of Theorem 2.46. Hence by a standard argument using the shadowing lemma, $H(p)$ is an isolated chain recurrence class. By Theorem 5 of [4], since M is connected, the homoclinic class $H(p)$ is in fact the whole manifold, hence f is Anosov. □

From [39] (or Corollary 4.25), we know that for C^1-generic diffeomorphisms far away from tangencies, a homoclinic class has a partially hyperbolic splitting with all central bundles dimension 1. We have the following result about the index of periodic orbits for Lyapunov stable homoclinic classes. It is a direct corollary of Corollary 4.26 and we omit the proof.

Corollary 4.28. *For generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathcal{HT}}$, if a homoclinic class $H(p)$ is Lyapunov stable and assume $T_{H(p)}M = E^s \oplus E^c_1 \oplus \cdots \oplus E^c_k \oplus E^u$ is the partially* *hyperbolic splitting, then the largest index of periodic points contained in $H(p)$ equals $d - \dim(E^s)$.*

4.5 Classification of non-hyperbolic homoclinic classes

In this section, we do some discussions related to Palis' conjecture and the local version of the generalized Palis' conjecture by Bonatti-Díaz (Conjecture 1). Recall that Palis' Conjecture claims the union of hyperbolic systems and systems with homoclinic tangencies or heterodimensional cycles are dense in $\mathrm{Diff}^1(M)$, and Bonatti-Díaz Conjecture states that the union of hyperbolic systems and systems with heterodimensional cycles are dense in $\mathrm{Diff}^1(M)$. Note that since on surfaces there is no heterodimensional cycle, Bonatti-Díaz Conjecture implies "Smale's" conjecture: hyperbolic systems are dense in the space of C^1 surface diffeomorphisms.

We discuss partial results known in this direction. The next two statements are consequences of the results in [17] and Theorem A.

Proposition 4.29. *For any generic $f \in \mathrm{Diff}^1(M)$, and for any non-hyperbolic homoclinic class $H(p)$ associated to a hyperbolic saddle p of index i, with $2 \leq i \leq d-2$, one of the following two possibilities holds:*

- *$H(p)$ contains a periodic point with different index,*
- *$H(p)$ has a partially hyperbolic splitting $E^s \oplus E^c$, $\dim(E^s) = i-1$, or $E^c \oplus E^u$, $\dim(E^c) = i+1$.*

Proof. We will assume that every periodic point contained in $H(p)$ has index i, since otherwise the first case in the statement holds. We consider the following two possibilities.

Assume that there is a dominated splitting $T_{H(p)}M = E \oplus F$ with $\dim(E) = i-1$, then by Corollary 4.24 and the fact that every periodic point contained in $H(p)$ has index i, the bundle E is contracted by Df, hence $E \oplus F$ is a partially hyperbolic splitting $E^s \oplus E^c$. Similarly if $\dim(E) = i+1$, we have that F is expanded by Df and $E \oplus F$ is a partially hyperbolic splitting $E^c \oplus E^u$. The second case stated in Proposition 4.29 holds.

Assume otherwise that $H(p)$ admits neither domination of index $i-1$ nor domination of index $i+1$. Since $H(p)$ is not hyperbolic, by Theorem D and the fact that every periodic point contained in $H(p)$ has index i, for any $\varepsilon > 0$, there is a periodic point $q \in H(p)$, such that $\chi_i(q) \in (-\varepsilon, 0)$ or $\chi_{i+1}(q) \in (0, \varepsilon)$. Then by Proposition 2.43, the diffeomorphism f can be C^1-approximated by diffeomorphisms with a heterodimensional cycle in the homoclinic class. By the items 5, 8 of Theorem 2.46, there is a periodic point of different index contained in $H(p)$, which contradicts the assumption. □

Proposition 4.30. *Assume* $\dim(M) = d \geq 3$. *For any generic* $f \in \mathrm{Diff}^1(M)$, *and for any non-hyperbolic homoclinic class* $H(p)$ *associated to a hyperbolic saddle* p *of index* $d-1$, *one of the following three possibilities holds:*

- $H(p)$ *contains a periodic point with different index,*
- $H(p)$ *has a partially hyperbolic splitting* $E^s \oplus E^c$, $\dim(E^s) = d-2$,
- $H(p)$ *is the Hausdorff limit of periodic sinks.*

Proof. We will assume that all periodic points contained in $H(p)$ have the same index $d-1$, otherwise the first case in the statement holds.

Assume that there is a dominated splitting $T_{H(p)}M = E \oplus F$ with $\dim(E) = d-2$, then by Corollary 4.24 and the fact that every periodic point contained in $H(p)$ has index $d-1$, the bundle E is contracted by Df, hence $E \oplus F$ is a partially hyperbolic splitting $E^s \oplus E^c$, which is the second case in the statement.

Assume otherwise that $H(p)$ admits no domination of index $d-2$. Since $H(p)$ is non-hyperbolic, by Theorem D and the fact that every periodic point contained in $H(p)$ has index $d-1$, there is a sequence of periodic points (q_n) contained in $H(p)$, such that $\chi_{d-1}(q_n) \to 0^-$ or $\chi_d(q_n) \to 0^+$ as $n \to +\infty$. Moreover, one can choose q_n to converge to $H(p)$ in the Hausdorff topology. If $\chi_{d-1}(q_n) \to 0^-$ occurs, then by Proposition 2.43 one can get a heterodimensional cycle in the homoclinic class by arbitrarily C^1-small perturbation. By the items 5, 8 of Theorem 2.46, there is a periodic point of different index contained in $H(p)$, which contradicts the assumption. If otherwise $\chi_d(q_n) \to 0^+$, then for any N and for any neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, there is $g \in \mathcal{U}$ and $n > N$, such that q_n is a periodic sink of g. By the item 9 of Theorem 2.46, there is a sequence of sinks converges to $H(p)$, which is the third case in the statement. $\square$

Proposition 4.30 has to be compared to a similar statement on surfaces:

Theorem (Pujals-Sambarino [64])**.** *For a generic surface diffeomorphism, any non-hyperbolic homoclinic class is the Hausdorff limit of periodic sinks or sources.*

Based on these propositions and on the previous known results [13, 16, 17, 20, 34, 38, 71], one can list the different possibilities of a non-hyperbolic homoclinic class and discuss Conjecture 1 in each case.

Non-hyperbolic homoclinic classes $H(p)$ for C^1-generic diffeomorphisms

Case a – *There exist two periodic points of different index.*
This is the case satisfied on examples and which corresponds to Conjecture 1.

Case b – *All periodic points have the same index i and the class $H(p)$ has a dominated splitting $T_{H(p)}M = E^s \oplus E^c_1 \oplus E^c_2 \oplus E^u$, with $\dim(E^c_j) \in \{0,1\}$, and $i = \dim(E^s \oplus E^c_1)$. Maybe E^s and/or E^u is trivial.*
This is exactly the case which occurs [34] when f is far from homoclinic tangencies and heterodimensional cycles, hence it is the case in the spirit of Palis conjecture.

Case c – *All periodic points have the same index i, the class $H(p)$ has a dominated splitting $T_{H(p)}M = E^s \oplus E^c \oplus E^u$, with $\dim(E^c) = 2$, and $\dim(E^s) = i-1$, the bundle E^c does not split, and there exist periodic points which contract and others which expand the volume along E^c. Maybe E^s and/or E^u is trivial.*
Periodic points which expand the volume along E^c are dense in the class, it is thus possible to turn them into points of index $i-1$. Since the class has only points of index i (even after perturbation), the strong stable manifold of such a periodic point has to intersect $H(p)$ only at the periodic point itself. One may then expect that for any point x in the class $W^{ss}(x) \cap H(p) = \{x\}$. In this case by [16] the class is contained in a submanifold tangent to $E^c \oplus E^u$. Arguing in a same way with periodic points which contract the volume along E^c, one deduces that $H(p)$ is contained in a locally invariant surface tangent to E^c. We are thus reduced to Smale's conjecture.

Case d – *All periodic points have the same index i, the class $H(p)$ has a dominated splitting $T_{H(p)}M = E \oplus E^u$, with $\dim(E) = i+1$, there is no dominated splitting corresponding to index i and along any periodic orbit the volume of planes in E is contracted (sectional dissipation in E).*
As in case (c), one can expect that the class is contained in a locally invariant submanifold tangent to E. We are thus reduced to the case of a homoclinic class whose periodic points have one-dimensional unstable spaces and sectional dissipative and has no domination corresponding to index $\dim(M) - 1$. We are thus reduced to a generalized Smale's conjecture for higher dimension, as described in [14, Conjecture 8].

Case d' – *Similar to case (d) but for f^{-1}.*
Here again, one may expect to reduce to the generalized Smale's conjecture.

We have the following result:

Proposition 4.31. *For a generic diffeomorphism $f \in \mathrm{Diff}^1(M)$, any non-hyperbolic homoclinic class has to satisfy one of the cases above.*

Proof. We consider a generic diffeomorphism $f \in \mathrm{Diff}^1(M)$ which satisfies Theorem 2.46, and a non-hyperbolic homoclinic class $H(p)$ associated to a hyperbolic periodic point p of index i. We assume that Case (a) does not occur, which means that all periodic points contained in $H(p)$ have the same index i. We consider the following possibilities.

Assume that there is a dominated splitting $T_{H(p)}M = E \oplus F$ with $\dim(E) = i$. If the bundle E is not uniformly contracted, then the bundle E has a dominated splitting $E^s \oplus E_1^c$ with $\dim(E^s) = i-1$. Otherwise, using Proposition 2.43 and Theorem D, one can get a heterodimensional cycle by arbitrarily C^1-small perturbation and by the items 5, 8 of Theorem 2.46, there is a periodic point of different index contained in $H(p)$, which contradicts the assumption. Moreover, by Corollary 4.26, the bundle E^s is uniformly contracted. Symmetrically, if the bundle F is not uniformly expanded, then it can be split as $E_2^c \oplus E^u$, where $\dim(E_2^c) = 1$ and E^u is uniformly expanded. Hence in this case, the homoclinic class has a partially hyperbolic splitting $T_{H(p)}M = E^s \oplus E_1^c \oplus E_2^c \oplus E^u$, with $\dim(E_j^c) \in \{0, 1\}$, and $i = \dim(E^s \oplus E_1^c)$. This is Case (b).

Assume now that the homoclinic class $H(p)$ admits no domination of index i. Consider the finest dominated splitting over $H(p)$. Combine all the contracted (center and expanded resp.) bundles and denote it by $E^s \oplus E^c \oplus E^u$, such that E^s and E^u are uniformly contracted and expanded bundles respectively and E^c is the center bundle. By Proposition 4.29 and Proposition 4.30, we have that $i = dim(E^s) + 1$ or $i = dim(E^s \oplus E^c) - 1$. We then consider the following two subcases.

If there exist both periodic points which contract and others which expand the volume along E^c, then there are both periodic orbits whose i^{th} exponent arbitrarily close to 0, and those whose $(i+1)^{th}$ exponent arbitrarily close to 0. Then the homoclinic class $H(p)$ admits both a domination of index $i-1$ and a domination of index $i+1$. Otherwise, using Proposition 2.43, one can get a heterodimensional cycle by arbitrarily C^1-small perturbation, and get a periodic point of different index in $H(p)$. Then we have that $T_{H(p)}M = E^s \oplus E^c \oplus E^u$, with $\dim(E^s) = i-1$ and $\dim(E^c) = 2$, and there is no finer dominated splitting along E^c. This is Case (c).

If there exist only periodic points which contract the volume along E^c, then arguing as in the previous case, the homoclinic class $H(p)$ admits a domination $E \oplus E^u$ where $\dim(E) = i+1$. This is Case (d). Symmetrically, if there exist only periodic points which expand the volume along E^c, then it is Case(d'). $\square$

Chapter 5

Non-hyperbolic ergodic measures on homoclinic classes

In this chapter, we give the proof of Theorem B, and we give a proposition that under some assumptions, one can get a non-hyperbolic ergodic measure whose support is the whole homoclinic class. We state Theorem B again below.

Theorem B. *For generic $f \in \mathrm{Diff}^1(M)$, a homoclinic class $H(p)$ which is not hyperbolic supports a non-hyperbolic ergodic measure μ.*

5.1 Reduction of Theorem B

Theorem B follows immediately from the following two theorems, considering whether the homoclinic class admits a dominated splitting corresponding to the index of p or not. The proofs of Theorem F and Theorem G are given in Section 5.2 and Section 5.3 respectively.

Theorem F. *For generic $f \in \mathrm{Diff}^1(M)$, if the homoclinic class $H(p)$ of a periodic point p of index i admits a dominated splitting $E \oplus F$ with $\dim E = i$, and if the bundle E is not uniformly contracted, then there exists an ergodic measure supported on $H(p)$ whose i^{th} Lyapunov exponent equals 0.*

Theorem G. *For generic $f \in \mathrm{Diff}^1(M)$, if the homoclinic class $H(p)$ of a periodic point p of index i does not admit a dominated splitting $E \oplus F$ with $\dim E = i$, then there exists a non-hyperbolic ergodic measure μ such that $\mathrm{supp}(\mu) = H(p)$. Moreover, if the i^{th} and $(i+1)^{th}$ Lyapunov exponents χ_i, χ_{i+1} of p satisfy $\chi_i + \chi_{i+1} < 0$, then the $(i+1)^{th}$ Lyapunov exponent of μ vanishes.*

5.2 The dominated case

In this section we complete the proof of Theorem F.

Lemma 5.1. *For any diffeomorphism $f \in \mathrm{Diff}^1(M)$, consider a non-trivial homoclinic class $H(p)$ having a dominated splitting $T_{H(p)}M = E \oplus F$ such that $\dim(E) = \mathrm{Ind}(p) = 1$. If the bundle E is not contracted, then there is an ergodic measure μ supported on $H(p)$, whose first Lyapunov exponent vanishes.*

Proof. Since the bundle E is not contracted, by Claim 1.7 of [34], there is an ergodic measure μ supported on $H(p)$, such that $\chi_1(\mu) \geq 0$, where $\chi_1(\mu)$ is the Lyapunov exponent of μ along the bundle E.

If $\chi_1(\mu) = 0$, then the ergodic measure μ which is supported on $H(p)$ is non-hyperbolic.

If $\chi_1(\mu) > 0$, then all Lyapunov exponents of μ are positive by the dominated splitting $E \oplus F$. Then μ is supported on a periodic source, which contradicts the fact that $\mathrm{supp}(\mu) \subset H(p)$ and that $H(p)$ is non-trivial. This finishes the proof of Lemma 5.1. □

Now will manage to prove Theorem F.

Proof of Theorem F. Now we consider a generic diffeomorphism $f \in \mathrm{Diff}^1(M)$ which satisfies Theorem 2.46, a hyperbolic periodic point p of index i, where $1 \leq i \leq d-1$, and a dominated splitting $T_{H(p)}M = E \oplus F$ where $\dim(E) = i$ and E is not contracted. By Theorem 2.44, we can assume that all periodic points contained in $H(p)$ have index larger than or equal to i, otherwise, there is an ergodic measure supported on $H(p)$ whose i^{th} Lyapunov exponent is zero and there is nothing need to prove.

If $\dim(E) = 1$, then the conclusion can be obtained from Lemma 5.1. Hence we can assume that $\dim(E) \geq 2$. We consider two subcases whether the bundle E has a dominated splitting $E_1 \oplus E_2$ with $\dim(E_2) = 1$ or not. Equivalently, we distinguish whether the homoclinic class $H(p)$ has a dominated splitting of index $i-1$ or not.

Case 1: $H(p)$ has a dominated splitting of index $i-1$. In this case, the bundle E has a dominated splitting into two bundles $E = E^s \oplus E^c$ such that $\dim(E^c) = 1$. By Corollary 4.24, the bundle E^s is contracted by Df. The bundle E^c is not contracted, since the bundle E is not contracted. By Claim 1.7 of [34], there is an ergodic measure μ supported on $H(p)$, such that $\chi_i(\mu) \geq 0$, where $\chi_i(\mu)$ is the Lyapunov exponent of μ along the bundle E^c.

If $\chi_i(\mu) = 0$, the conclusion of Theorem F holds.

If $\chi_i(\mu) > 0$, then μ is a hyperbolic measure because the bundle E^s is contracted by Df. Moreover, the non-uniform hyperbolic splitting of μ is a dominated spitting $E^s \oplus (E^c \oplus F)$. By Proposition 2.45, there is a hyperbolic periodic point q of index $i-1$, such that $\mathrm{supp}(\mu) \subset H(q)$. By the item 2 of Theorem 2.46, q belongs to $H(p)$, which contradicts the assumption that all periodic points contained in $H(p)$ have index larger than or equal to i.

Case 2: $H(p)$ has no dominated splitting of index $i-1$. By Theorem D, since the bundle E is not contracted, for any $\varepsilon > 0$, there is a periodic point p_ε homoclinically related to p, such that $\chi_i(p_\varepsilon) \in (-\varepsilon, 0)$. Since $H(p)$ has no dominated splitting of index $i-1$, by Proposition 2.43, there is an arbitrarily small perturbation g of f, such that $H(p_g, g)$ has a heterodimensional cycle associated to $\mathrm{Orb}(p_g)$ and $\mathrm{Orb}(q_g)$ with $\mathrm{Ind}(q_g) = i-1$. Then by the item 8 of Theorem 2.46, there is a periodic point $q \in H(p)$ whose index equals $i-1$, which contradicts the assumption that all periodic points contained in $H(p)$ have index larger than or equal to i.

The proof of Theorem F is now complete. $\square$

5.3 The non-dominated case

In this section we prove Theorem G.

5.3.1 Multiple almost shadowing of $\mathrm{Orb}(p)$ with a weak Lyapunov exponent

The following proposition states that, for generic $f \in \mathrm{Diff}^1(M)$, if a homoclinic class $H(p)$ has no dominated splitting of index $\mathrm{Ind}(p)$, then by a C^1-small perturbation, arbitrarily dense in $H(p)$, there is a periodic orbit that multiple almost shadows the orbit of p and that has a Lyapunov exponent close to 0.

Proposition 5.2. *For generic $f \in \mathrm{Diff}^1(M)$, consider a center-dissipative hyperbolic periodic saddle p of index i which has simple spectrum. Assume that the homoclinic class $H(p,f)$ has no dominated splitting of index i. Then for any $\varepsilon, \gamma > 0$, for any $\varkappa \in (0,1)$, and for any C^1-neighborhood $\mathcal{U}$ of f, there are a diffeomorphism $g \in \mathcal{U}$, and a hyperbolic saddle q of f, such that:*

1. *the saddle q is homoclinically related to p with respect to f and g,*
2. *the orbit of p is $(\gamma, \varkappa)$-multiple almost shadowed by the orbit of q,*
3. *the Hausdorff distance between $\mathrm{Orb}(q)$ and $H(p,f)$ is less than ε,*
4. *g coincides with f on $\mathrm{Orb}(q)$ and outside a small neighborhood of $\mathrm{Orb}(q)$,*
5. *the saddle q has simple spectrum with respect to g,*
6. *$\chi_{i+1}(q,g) \in (0,\varepsilon)$.*

Proof. We assume that f satisfies the properties stated in Lemma 2.39 and Theorem 2.46. Without loss of generality, we can assume that $\varepsilon > 0$ is small such that any diffeomorphism h that is ε-close to f in $\mathrm{Diff}^1(M)$ is contained in $\mathcal{U}$ and such that $\varepsilon < |\chi_j|$, where $j = i, i+1$. Then there are two positive integer T and τ that satisfy the conclusion of Lemma 2.41 associated to the constant ε.

By the definition of dominated splitting, there is $\eta > 0$, such that, any invariant compact set that is η-close to $H(p)$ in the Hausdorff distance has no T-dominated splitting of index i. Moreover, we can assume that $\eta < \varepsilon$.

By Lemma 2.39, there is a center-dissipative periodic saddle $\mathrm{Orb}(q)$ with simple spectrum homoclinically related to p such that,

– the orbit of p is $(\gamma, \varkappa)$-multiple almost shadowed by the orbit of q,

– the Hausdorff distance between $\mathrm{Orb}(q)$ and $H(p, f)$ is less than η,

– the Lyapunov exponents of $\mathrm{Orb}(q)$ are close to those of $\mathrm{Orb}(p)$.

Hence the items 2, 3 are satisfied. Moreover, the periodic point q can be chosen such that its period $\pi(q)$ is larger than τ.

Now we do a perturbation to get a diffeomorphism g that satisfies the item 1, 4, 5, 6.

Consider the hyperbolic periodic orbit $\mathrm{Orb}(q, f)$. By the choice of η, one can see that $\mathrm{Orb}(q)$ has no T-dominated splitting of index i.

By Lemma 2.41, for each $n = 0, 1, \cdots, \pi(q) - 1$, there is a one-parameter family of matrices $(A_{n,t})_{t\in[0,1]}$ in $GL(d, \mathbb{R})$, such that, denote by $B_t = A_{\pi(q)-1,t} \circ \cdots \circ A_{0,t}$ for $t \in [0, 1]$, the following properties are satisfied.

– $A_{n,0} = Df(f^n(q))$,

– $\|A_{n,t} - Df(f^n(q))\| < \varepsilon$ and $\|A^{-1}_{n,t} - Df^{-1}(f^n(q))\| < \varepsilon$, for any $t \in [0, 1]$,

– $\chi_{i+1} \in (0, \varepsilon)$,

– $\chi_j(B_t) = \chi_j(B_0)$, for any $t \in [0, 1]$ and for any $j \in \{1, 2, \cdots, d\}\setminus\{i, i+1\}$,

– B_t is hyperbolic for any $t \in [0, 1]$,

Take a small constant $\delta > 0$, such that the local manifolds $W^s_\delta(\mathrm{Orb}(q), f)$ and $W^u_\delta(\mathrm{Orb}(q), f)$ of size δ are two embedding sub-manifolds of dimension i and $d - i$ respectively. Then there are two transverse homoclinic points $z \in W^s_\delta(\mathrm{Orb}(q), f) \pitchfork W^u(p, f)$ and $y \in W^u_\delta(\mathrm{Orb}(q), f) \pitchfork W^s(p, f)$ since the periodic orbit $\mathrm{Orb}(q)$ is homoclinically related to p with respect to f. Consider the two small compact sets $\{z\}$ and $\{y\}$ as K^s and K^u. There is a small neighborhood V of $\mathrm{Orb}(q)$, such that $\overline{V}$ is disjoint with $\mathrm{Orb}^-(z, f)$, $\mathrm{Orb}^+(y, f)$ and $\mathrm{Orb}(p)$. By Lemma 2.42, there is a diffeomorphism g that is ε-close to f in $\mathrm{Diff}^1(M)$, and that satisfies the following properties:

a). g coincides with f on the orbit of $\mathrm{Orb}(q)$ and outside V;

b). $z \in W^s_\delta(\mathrm{Orb}(q), g)$ and $y \in W^u_\delta(\mathrm{Orb}(q), g)$;

c). $Dg(g^n(q)) = Dg(f^n(q)) = A_{n,1}$ for all $n = 0, \cdots, \pi(q) - 1$.

Then we have that $z \in W^s(\mathrm{Orb}(q), g) \cap W^u(p, g)$ and $y \in W^u(\mathrm{Orb}(q), g) \cap W^s(p, g)$, and by an arbitrarily small C^1-perturbation if necessary, we can assume that the intersections are transverse. Hence $\mathrm{Orb}(q)$ is still homoclinically related to p under g, which is the item 1 in Proposition 5.2. The item 4 is automatically satisfied by the item a. The items 5, 6 are satisfied by the item c and the properties of the one-parameter families $(A_{n,t})_{t\in[0,1];n=0,1,\cdots,\pi(q)-1}$.

The proof of Proposition 5.2 is now complete. □

5.3.2 Construction of sequences of weak periodic orbits

The following proposition gives a sequence of periodic orbits that have some shadowing properties for C^1-generic diffeomorphisms.

Proposition 5.3. *For generic $f \in \mathrm{Diff}^1(M)$, assume that p is a center-dissipative hyperbolic saddle of index i with simple spectrum whose homoclinic class $H(p)$ has no dominated splitting of index i. Then, there is a sequence of center-dissipative periodic points $(q_n)_{n\geq 1}$ with simple spectrum, together with a sequence of positive numbers $(\gamma_n)_{n\geq 1}$, such that, for any $n \geq 1$, the followings are satisfied.*

1. *q_n is homoclinically related to p.*
2. *$\mathrm{Orb}(q_n, f)$ is $\frac{1}{4^n}$-dense in $H(p)$.*
3. *$\gamma_n < \frac{1}{2}\gamma_{n-1}$ and the orbit of q_{n-1} is $(\gamma_{n-1}, 1 - \frac{1}{2^{n-1}})$-multiple almost shadowed by the orbit of q_n.*
4. *There exists a positive integer $N_n > \pi(q_n)$, such that for any point x contained in the $2\gamma_n$-neighborhood of $\mathrm{Orb}(q_n)$, we have $L^{(N_n)}_{d-i}(x) - L^{(N_n)}_{d-i-1}(x) \in (0, \frac{1}{2^n})$.*

Proof. Since f is a C^1-generic diffeomorphism, by the item 4 of Theorem 2.46, for a diffeomorphism g close to f in $\mathrm{Diff}^1(M)$, the homoclinic class $H(p_g, g)$ is close to $H(p, f)$ in the Hausdorff topology. Hence one can see that the items 1, 2, 3, 5, 6 in Proposition 5.2 are persistent under C^1-perturbations. Therefore by a standard Baire argument, the following statement holds.

For generic $f \in \mathrm{Diff}^1(M)$, assume p is a center-dissipative hyperbolic saddle of index i which has simple spectrum, if the homoclinic class $H(p)$ has no dominated splitting of index i, then for any $\varepsilon, \gamma > 0$, and any $\varkappa \in (0, 1)$, there is a center-dissipative periodic saddle q with simple spectrum homoclinically related to p, such that:

– $\mathrm{Orb}(p)$ *is $(\gamma, \varkappa)$-multiple almost shadowed by* $\mathrm{Orb}(q)$,

– $\mathrm{Orb}(q)$ *is* ε*-dense in* $H(p)$,

– $\chi_{i+1}(q,f) \in (0,\varepsilon)$.

One may assume that the diffeomorphism f in the statement of Proposition 5.3 satisfies the property above and the properties of Lemma 2.39 and Theorem 2.46. Now we construct the sequence of periodic orbits. To make it complete, we take $q_0 = p$ and $\gamma_0 = 1$.

Assume $\mathrm{Orb}(q_n)$ and γ_n have been taken to satisfy the properties stated in the proposition for any $n \leq k-1$. We construct $\mathrm{Orb}(q_k)$, γ_k and N_k. We have that $H(q_{k-1}) = H(p)$. Consider the periodic point q_{k-1}, since $H(q_{k-1})$ has no domination of index i, and by the choice of $\mathcal{R}$, there is a periodic point q_k with simple spectrum homoclinically related to q_{k-1}, such that, $\mathrm{Orb}(q_{k-1})$ is $(\gamma_{k-1}, 1-\frac{1}{2^{k-1}})$-multiple almost shadowed by $\mathrm{Orb}(q_k)$, $\mathrm{Orb}(q_k)$ is $\frac{1}{4^k}$-dense in $H(p)$, and $\chi_{i+1}(q_k,f) \in (0,\frac{1}{4^k})$. Hence the items 1, 2, 3 are satisfied. By the fact that $\chi_{i+1}(\mu,f) = \lim_{m\to+\infty}(L^{(m)}_{d-i}(x,f) - L^{(m)}_{d-i-1}(x,f))$, there is $N_k > \pi(q_k)$, such that for any $x \in \mathrm{Orb}(q_k)$, we have $L^{(N_k)}_{d-i}(x) - L^{(N_k)}_{d-i-1}(x) \in (0,\frac{1}{2^k})$. Then there is a constant $\gamma_k \in (0,\frac{\gamma_{k-1}}{2})$, such that for any x contained in the $2\gamma_k$-neighborhood of $\mathrm{Orb}(q_k)$, we have that $L^{(N_k)}_{d-i}(x) - L^{(N_k)}_{d-i-1}(x) \in (0,\frac{1}{2^k})$. Then the item 4 is satisfied.

The proof of Proposition 5.3 is now complete. $\square$

5.3.3 End of the proof of Theorem G

Now we can prove Theorem G. By the item 1 and 10 of Theorem 2.46, we can assume that the periodic point p is center-dissipative and has simple spectrum. Then there is a sequence of center-dissipative hyperbolic periodic orbits (q_n) that satisfies the properties in Proposition 5.3.

By Lemma 2.40, denoting by μ_n the probability atomic measure uniformly distributed on the orbit $\mathrm{Orb}(q_n)$ for each n, the weak-$*$-limit of μ_n is an ergodic measure μ, whose support is:

$$\mathrm{supp}(\mu) = \bigcap_{n=1}^{\infty}(\overline{\bigcup_{k=n}^{\infty}\mathrm{Orb}(q_n)}) = H(p).$$

It only remains to show that μ is a non-hyperbolic measure, which is from the following claim.

Claim 5.4. *For the ergodic measure* μ, *we have that* $\chi_{i+1}(\mu,f) = 0$.

Proof. Since the orbit of q_n is a $(\gamma_n, 1-\frac{1}{2^n})$-multiple almost shadowed by the orbit of q_{n+1}, there are a subset $\Gamma_n \subset \mathrm{Orb}(q_n)$ and a map $\rho_n : \Gamma_n \mapsto \mathrm{Orb}(q_{n-1})$ for each $n \geq 2$ from Definition 2.38. Take $Y_n = \rho_n^{-1} \circ \cdots \circ \rho_2^{-1}(\mathrm{Orb}(q_1))$, we can see that Y_n is well-defined. Take the upper topological limit

$$Y = \limsup_{n\to+\infty} Y_n.$$

Since Y is a compact set and μ is the limit measure of μ_n, we have that

$$\mu(Y) \geq \limsup_{n\to+\infty} \mu_n(Y_n) \geq \prod_{k=1}^{n-1}(1-\frac{1}{2^n}) > 0.$$

By the fact that $\gamma_{n+1} < \frac{1}{2}\gamma_n$, we can see that the set Y is contained in the $2\gamma_n$-neighborhood of $\mathrm{Orb}(q_n)$ for every $n \geq 1$. Then for any $x \in Y$, and any $n \geq 1$, we have that $L_{d-i}^{(N_n)}(x) - L_{d-i-1}^{(N_n)}(x) \in (0, \frac{1}{2^n})$ for the strictly increasing sequence $(N_n)_{n\geq 1}$. By the facts that μ is ergodic and $\mu(Y) > 0$, we have that for μ-a.e. $x \in Y$,

$$\begin{aligned}\chi_{i+1}(\mu, f) &= \lim_{m\to+\infty}(L_{d-i}^{(m)}(x,f) - L_{d-i-1}^{(m)}(x,f))\\ &= \lim_{n\to+\infty}(L_{d-i}^{(N_n)}(x,f) - L_{d-i-1}^{(N_n)}(x,f))\\ &= 0.\end{aligned}$$

□

5.4 Non-hyperbolic ergodic measures with full support

We state a result in this section, which shows that one can obtain a non-hyperbolic measure with full support under certain assumptions. This generalizes a previous result of [22].

Proposition 5.5. *For generic $f \in \mathrm{Diff}^1(M)$, if a homoclinic class $H(p)$ contains periodic points with different indices, then there is a non-hyperbolic ergodic measure μ with* $\mathrm{supp}(\mu) = H(p)$.

Proof. By the items 2, 5 of Theorem 2.46, we can assume that there is a periodic point q such that $H(q) = H(p)$, and p, q have indices $i, i+1$ respectively. We consider the following two cases.

If $H(p)$ has no dominated splitting of index i or of index $i+1$, then by Theorem G, there is a non-hyperbolic ergodic measure μ such that $\mathrm{supp}(\mu) = H(p)$.

If otherwise, the homoclinic class $H(p)$ has both a dominated splitting of index i and a dominated splitting of index $i+1$, then by Theorem 2.44, there is a non-hyperbolic ergodic measure μ such that $\mathrm{supp}(\mu) = H(p)$. □

Chapter 6

Dominated splitting on Lyapunov stable aperiodic classes

In this chapter, we give the proof of Theorem C, which we restate below.

Theorem C. *For generic $f \in \mathrm{Diff}^1(M)$, if a Lyapunov stable aperiodic class Λ of f admits a dominated splitting $T_\Lambda M = E \oplus F$, then one (and only one) of the following two cases holds: E is contracted or F is expanded.*

6.1 The predefined settings

First, we take $f \in \mathcal{R}$, where $\mathcal{R}$ is a residual subset in $\mathrm{Diff}^1(M)$ that satisfies the conclusions of Theorem 2.46 and the properties stated in Proposition 3.3. Later we will assume also that f belongs to another residual subset $\mathcal{R}_1$ of $\mathrm{Diff}^1(M)$, which will be defined below. We consider a Lyapunov stable aperiodic class Λ of f admitting a dominated splitting $T_\Lambda M = E \oplus F$.

By taking an adapted metric [43], we assume that $T_\Lambda M = E \oplus F$ is a $(1, \lambda^2)$-dominated splitting for a constant $0 < \lambda < 1$. To simplify the notations, we call $E \oplus F$ a λ^2-dominated splitting. By Remark 2.8 we can take a neighborhood V_0 of Λ and a C^1-neighborhood $\mathcal{U}_0$ of f, such that, for any $g \in \mathcal{U}_0$, the invariant compact set $\bigcap_{n \in \mathbb{Z}} g^n(\overline{V_0})$ has a λ^2-dominated splitting that is the extension of $T_\Lambda M = E \oplus F$. To simplify the notations, we still denote this dominated splitting by $E \oplus F$.

To prove Theorem C, we assume that neither $E|_\Lambda$ is contracted nor $F|_\Lambda$ is expanded. Then we show that one can obtain a periodic orbit that intersects the aperiodic class Λ, which leads a contradiction. Take two numbers $\lambda_1 < \lambda_2$ in the interval $(\lambda, 1)$.

6.2 Existence of a bi-Pliss point whose ω-limit set is E contracted

Similar to Lemma 4.6, we have the following lemma.

Lemma 6.1. *There is a λ_2-bi-Pliss point $x \in \Lambda$, such that $\omega(x)$ is a λ_1-F-weak set and $x \notin \omega(x)$.*

Lemma 4.6 is for homoclinic classes and for the assumption that the bundle E is not contracted. By considering the diffeomorphism f^{-1} and the bundle F, one can see that Lemma 6.1 follows exactly the same arguments. Hence we do not give the proof here.

6.3 Existence of E-contracted periodic orbits by perturbations

We take the λ_2-bi-Pliss point $x \in \Lambda$ from Lemma 6.1. Then we have that $\omega(x)$ is a λ_1-F-weak set and $x \notin \omega(x)$. We have the following lemma to get E-uniformly contracted periodic orbits close to Λ by C^1-small perturbations.

Lemma 6.2. *For any C^1-neighborhood $\mathcal{U}$ of f, any neighborhood V of Λ, and any neighborhood U_x of x, there are a diffeomorphism $g \in \mathcal{U}$ and a periodic point $q \in U_x$ of g with period τ, such that* $\mathrm{Orb}(q,g) \subset V$*, and*

$$\prod_{i=0}^{\tau-1} \|Dg|_{E(g^i(q))}\| < {\lambda_1}^{\tau}.$$

Proof. Take a C^1-neighborhood $\mathcal{U}$ of f, a neighborhood V of Λ and a neighborhood U_x of x. Without loss of generality, we assume that $\mathcal{U} \subset \mathcal{U}_0$, $V \subset V_0$ and $U_x \subset V_0$.

By Lemma 2.19 and the compactness of $\omega(x)$, there is an integer $T \geq 1$, such that, for any $y \in \omega(x)$,

$$\prod_{i=0}^{T-1} \|Df|_{E(f^i y)}\| < \lambda^T.$$

Then there are a neighborhood $U \subset V$ of $\omega(x)$, and a C^1-neighborhood $\mathcal{V} \subset \mathcal{U}$ of f, such that, for any point y with $\mathrm{Orb}(y,g) \subset V$, if $g^i(y) \subset U$ for any $0 \leq i \leq T$, then it holds that

$$\prod_{i=0}^{T-1} \|Dg|_{E(g^i y)}\| < \lambda^T.$$

Take $C = sup\{\|Dg\| : g \in \mathcal{V}\}$. Take a small neighborhood $U_x \subset V$ of x. Considering the diffeomorphism f^{-1}, by Proposition 3.3, there is an integer L associated to $(f, \mathcal{V}, U, U_x, V)$. Take an integer $k \in \mathbb{N}$, such that $\lambda^{nT} \cdot C^{LT+T} < \lambda_1^{nT+LT+T}$ for any $n \geq k$. Take $m = kT + LT + T$, then by Proposition 3.3, there are a diffeomorphism $g \in \mathcal{V} \subset \mathcal{U}$ and a periodic point $q \in U_x$ of g, such that $\mathrm{Orb}(q, g) \subset V$ satisfies $\#(\mathrm{Orb}(p, g) \cap U) \geq m$, and $\#(\mathrm{Orb}(p, g) \setminus U) \leq L$. Denote by τ the period of q under the iterate of g. It can be written as $\tau = nT + LT + r$, where $0 \leq r < T$. Then by the distribution of the points of $\mathrm{Orb}(q, g)$, there are at least n pieces of segments $\{f^{l_i}(q), f^{l_i+1}(q), \cdots, f^{l_i+T-1}(q)\}_{1 \leq i \leq n}$ that are pairwise disjoint and contained in U. Hence we have that

$$\prod_{i=0}^{\tau-1} \|Dg|_{E(g^i(q))}\| \leq C^{LT+T} \cdot \prod_{j=1}^{n} \prod_{i=l_j}^{l_j+T-1} \|Dg|_{E(g^i(q))}\|$$
$$\leq \lambda^{nT} \cdot N^{LT+T} < {\lambda_1}^{nT+LT+T} < {\lambda_1}^{\tau}.$$

This finishes the proof of Lemma 6.2. □

6.4 Generic existence of E contracted periodic orbits

From Lemma 6.2, we obtain some E contracted periodic orbits close to the aperiodic class Λ that has a point close to the λ_2-bi-Pliss point $x \in \Lambda$ by C^1-small perturbations of f. Then, with a standard Baire argument (see for example [44]), we can obtain such periodic orbits for the generic diffeomorphism f itself.

Lemma 6.3. *There is a residual subset $\mathcal{R}_1 \subset \mathrm{Diff}^1(M)$, such that, if $f \in \mathcal{R}_1$, then for any neighborhood V of Λ, and any neighborhood U_x of the λ_2-bi-Pliss point x, there is a periodic point $q \in U_x$ of f with period τ, such that* $\mathrm{Orb}(q) \subset V$, *and*

$$\prod_{i=0}^{\tau-1} \|Df|_{E(f^i(q))}\| \leq {\lambda_1}^{\tau}.$$

Proof. Take a countable basis $(U_m)_{m\geq 1}$ of M. Denote by $(V_n)_{n\geq 1}$ the countably collection of sets such that each V_n is a union of finitely many sets of the basis $(U_m)_{m\geq 1}$.

Let $\mathcal{H}_{m,n,j}$ be the set of C^1 diffeomorphisms h, satisfying the following properties.

There is a hyperbolic periodic orbit $\mathrm{Orb}(q, h')$*, such that*

- $\mathrm{Orb}(q, h') \subset V_n$ *and* $\mathrm{Orb}(q, h') \cap U_m \neq \emptyset$*,*

- *there is a dominated splitting* $T_{\mathrm{Orb}(q,h')}M = E \oplus F$ *with* $\dim(E) = j$*, and, denoting by* τ *the period of* q*, then*

$$\prod_{i=0}^{\tau-1} \|Dh'|_{E(h'^i(q))}\| < \lambda_1{}^{\tau}.$$

Notice that $\mathcal{H}_{m,n,j}$ is an open subset of $\mathrm{Diff}^1(M)$. Take $\mathcal{N}_{m,n,j} = \mathrm{Diff}^1(M) \setminus \overline{\mathcal{U}_{m,n,j}}$, then the set $\mathcal{H}_{m,n,j} \cup \mathcal{N}_{m,n,j}$ is an open and dense subset of $\mathrm{Diff}^1(M)$.

Let

$$\mathcal{R}_1 = \mathcal{R} \cap (\textstyle\bigcap_{m,n\geq 1, 1\leq j\leq d-1} (\mathcal{H}_{m,n,j} \cup \mathcal{N}_{m,n,j})),$$

where $\mathcal{R}$ is taken from Theorem 2.46. Then the set $\mathcal{R}_1$ is a residual subset of $\mathrm{Diff}^1(M)$. We now prove that the conclusion of Lemma 6.3 is valid for the residual subset $\mathcal{R}_1$.

Take any diffeomorphism $f \in \mathcal{R}_1$. For any neighborhood V of Λ, and any neighborhood U_x of x, there are two integers m and n, such that $V_n \subset V$ and $x \in U_m \subset U_x$. By Lemma 6.2, for any neighborhood $\mathcal{V}$ of f, there is a diffeomorphism $g \in \mathcal{V}$ such that $g \in \mathcal{H}_{m,n,j}$, where $j = \dim(E)$. This means that $f \in \overline{\mathcal{H}_{m,n,j}}$. Then $f \notin \mathcal{N}_{m,n,j}$, and thus $f \in \mathcal{H}_{m,n,j}$. The proof of Lemma 6.3 is finished by the construction of $\mathcal{H}_{m,n,j}$. □

6.5 Proof of Theorem C

We have the following lemma, which shows that under certain conditions, there is a point whose unstable manifold touches the stable manifold of a periodic orbit.

Lemma 6.4. *Given three constants* $0 < \lambda < \lambda_1 < \lambda_2 < 1$*. For a diffeomorphism* $f \in \mathrm{Diff}^1(M)$*, consider an invariant compact set* K *admitting a* $(1, \lambda^2)$*-dominated splitting* $T_K M = E \oplus F$*. Assume that* $x \in K$ *is a* λ_2*-bi-Pliss point. Assume that for any neighborhood* V *of* K*, and any neighborhood* U_x *of* x*, there is a periodic point* $q \in U_x$ *of* f *with period* τ*, such that* $\mathrm{Orb}(q) \subset V$*, and*

$$\prod_{i=0}^{\tau-1} \|Df|_{E(f^i(q))}\| \leq \lambda_1{}^{\tau}.$$

Then there is a point $y \in K$ *and a periodic point* q *of* f*, such that* $W^u(y) \cap W^s(q) \neq \emptyset$*.*

Proof. By the assumption, there is a sequence of periodic points $\{q_n\}_{n\geq 1}$ which converges to x such that $\mathrm{Orb}(q_n)$ accumulates to a subset of K, and

$$\prod_{i=0}^{\tau_n-1} \|Df|_{E(f^i(q_n))}\| < \lambda_1{}^{\tau},$$

where τ_n is the period of q_n, for any $n \geq 1$.

By the Pliss Lemma, there are λ_1-E-Pliss points on $\mathrm{Orb}(q_n)$. Consider all pairs of consecutive λ_1-E-Pliss points $(f^{k_i^n}(q_n), f^{l_i^n}(q_n))_{1\leq i\leq m_n}$ on $\mathrm{Orb}(q_n)$. We consider whether the sequence of numbers $(l_i^n - k_i^n)_{n\geq 1, 1\leq i\leq m_n}$ is uniformly bounded or not.

Case 1. If there is a number N such that $0 < l_i^n - k_i^n \leq N$, for all i and all $n \geq 1$, then the set $\overline{\bigcup_{n\geq 1}\mathrm{Orb}(q_n)}$ is an E-contracted set. Hence any q_n has a uniform stable manifold with dimension $\dim(E)$ by Remark 2.17. Since x is a λ_2-bi-Pliss point, when q_n is close enough to x, we have that $W^u(x) \cap W^s(q_n) \neq \emptyset$. Then we take $y = x$ and $q = q_n$.

Case 2. We consider the case where the sequence $(l_i^n - k_i^n)_{n\geq 1, 1\leq i\leq m_n}$ is not uniformly bounded. By considering a subsequence if necessary and to simplify the notations, we assume that the sequence of consecutive λ_1-E-Pliss points $(f^{k_1^n}(q_n), f^{l_1^n}(q_n))$ satisfies $l_1^n - k_1^n \to +\infty$ as $n \to +\infty$. By item 1 of Lemma 2.22, any limit point $y \in \Lambda$ of the sequence $\{f^{l_1^n}(q_n)\}$ is a λ_1-bi-Pliss point. Hence when $f^{l_1^n}(q_n)$ is close enough to y, we have that $W^u(y) \cap W^s(f^{l_1^n}(q_n)) \neq \emptyset$. Then we take $q = f^{l_1^n}(q_n)$.

The proof of Lemma 6.4 is now completed. □

Let $\mathcal{R}_0 = \{f \in \mathcal{R} \cap \mathcal{R}_1 :$ and f satisfies Proposition 3.3$\}$. Then $\mathcal{R}_0$ is a residual subset of $\mathrm{Diff}^1(M)$. We prove that Theorem C is satisfied for all the diffeomorphisms in $\mathcal{R}_0$.

End of the proof of Theorem C. We prove Theorem C by contradiction. If neither E is contracted nor F is expanded, then by Lemma 6.4, we have that $W^u(y) \cap W^s(q) \neq \emptyset$ for some $y \in \Lambda$ and some periodic point q. Since Λ is a Lyapunov stable aperiodic class, we have that $W^u(y) \subset \Lambda$, hence $q \in \Lambda$, which contradicts the fact that Λ contains no periodic point. This concludes Theorem C. □

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Acknowledgements

First I would like to thank China Scholarship Council (CSC) for financial support (201306010008) during my two years in France.

I am very grateful to my supervisors Prof. Lan Wen at Peking University and Prof. Sylvain Crovisier at University Paris-Sud 11. This thesis can not be completed without you. Lan has taken me into the field of Dynamical Systems and I have learned a lot from him, not only in mathematics but also in how to do things. To us students, Lan has always been very thoughtful, generous and attentive. He has always encouraged me and has spared no effort to lead me in the research. I first met Sylvain when he visited Peking University in 2012. At that time, I have already admired him as a kind person and a brilliant mathematician. I went to Orsay in 2013 and spent two years there under the supervision of Sylvain. He has dedicated a lot of time and has given me a lot of help not only in my research but also in the daily life of me and my wife. I would like to thank his infinite patience and dedication. Both Lan and Sylvain are nice persons and good supervisors who can make you feel very comfortable talking with them and who are always very encouraging. They has always been and will always be a role model in my whole life.

I would like to thank the professors at Peking University: Shaobo Gan, Wenxiang Sun and Zhihong Xia. They have given me many help not only in my research but also in my daily life. I have learned a lot from them in the classes and in the seminars. I have admired their enthusiasm and profound knowledge in mathematics. I appreciate Martín Sambarino and Marie-Claude Arnaud for refereeing my thesis.

I would like to thank Yongluo Cao, Dawei Yang, Xiao Wen, Martín Sambarino, Rafael Potrie, Nicolas Gourmelon, Katstutoshi Shinohara, for the discussions and useful comments when I wrote my papers. I would also like to thank Meirong Zhang, Jiazhong Yang, Shengzhi Zhu, Ming Li, Yong Zhang, Baolin He, Chao Liang, Gang Liao, Xueting Tian, Sheng Qian, Fang Wang, Peng Sun, Pengfei Zhang, Yongxia Hua, Fei Liu, Christian Bonatti, Jérôme Buzzi, Ali Tahzibi, Rémi Leclercq, Frédéric Paulin, Thierry Bousch. Also I would like to thank my classmates and friends at Peking University: Yi Shi, Yan Ren, Rusong Zheng, Yun Yang, Shirou Wang, Weisheng Wu, Jinhua Zhang, Cheng Cheng, Xiankun Ren, Xufeng Guo, Sixu Liu, Qianying Xiao, Longmei Shu, Changwu Zou, Xiaolong Li. I would like to thank the whole

Dynamical Systems Group at Peking University and at University Paris-Sud 11. I have learned many things from you during the discussions and seminars in the past years.

I would like to thank my friends: Can Zhang, Feng Xing, Jian Xiao, Bo Xia, Zheng Chen, Yi Huang, Haiyan Xu, Weihua He, Çağri Sert, Weikun He, Shu Shen. Your company with me and my wife has made our stay in France much happier.

I am also appreciated Mme. Valérie Lavigne for all the help during my stay at Orsay.

Of course I would like to thank my family: my parents and my wife Hongshu Gao. Thank you for always supporting me and accompanying me.

Publications

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